Participating Life Insurance Contracts with Periodic Premium Payments

This version: February 15, 2019

Abstract

We consider participating life insurance products and their benefits to the insured. Our main focus is on the impact of different contribution schemes, i.e. when and how much the insured contributes. This is important since the insured’s benefits depend on the performance of the investment strategy conducted by the product provider. In addition, we consider the interactions of the premium contribution scheme, embedded guarantees, and different management rules accounting of a regime switch in the risk profile of the insurance company. We shed light on two effects on the utility of the insured. The combination of contribution scheme, embedded guarantees and management rule has an impact on the investment risk of the insured. A second main effect is a price effect. Assuming that the guarantees are fairly priced, the guarantee costs also depend on all the above mentioned factors.

Keywords: Guarantee schemes, life insurance, periodic payments, investment strategies, expected utility, management rule

JEL: G13, G22
1 Introduction

A common practice in the life insurance sector is that the insured participates with her saving portion (i.e. the contributions of the insured minus the insurance costs) in the investment results of the insurance company. Normally, this participation only refers to a possible surplus. Regulatory rules and the competition imply that the insured receives at least some guaranteed value or a guaranteed interest on her savings amounts. The contributions of the insured can be given in terms of an upfront contribution or ongoing, e.g. yearly, premium payments. Since different contribution schemes have an impact on when and how the insured is invested in the asset side of the insurance company, they also imply different guarantee costs and benefits to the insured.

While there exists a large strand of literature focusing on participating life insurance contracts where implicitly an upfront premium payment is assumed, an analysis of the impact of different premium contribution schemes on the guarantee price and expected utility of the insured is to our knowledge still missing.

We fill this gap by analyzing and comparing periodic payment schemes with respect to the implied guarantee costs and the expected utility to the insured. Furthermore, we discuss the interaction between guarantee products and management rules concerning the investment decisions (the risk profile) of the insurance company. The results give insights for both, practitioners and academics in the research field of life insurance contracts and risk management.

Our proceeding is as follows. We assume a Black and Scholes type model setup consisting of two assets, a risky stock (or index) and a risk free asset growing with constant interest rate. The price dynamics of the risky asset is driven by a geometric Brownian motion. In a benchmark scenario where the insurance company invests a constant fraction of the asset side in the risky asset, the asset side of the insurance company is also driven by a geometric Brownian motion. Here, the riskiness of the investments can exclusively be described by the volatility of the risky asset scaled by the constant investment fraction, i.e. the risk structure is constant over the investment horizon. We also add management rules which, at discrete points in time, may imply a regime switch, i.e. a different risk profile of the asset side.

The present value of the contributions of the insured are normalized to one. However, she can decide between different contribution schemes in terms of the premium contributions. Assuming that the risk preferences of the investor are described by a constant relative risk aversion (CRRA), we consider her portfolio planning problem in terms of maximizing her expected utility w.r.t. how to split her contributions over time.

The solution in the benchmark case without a regime switch can, to a large extent, be explained by means of the classic Merton problem with and without an additional
constraint (posed by the guarantee) on the payoff value. The optimal contribution scheme (i.e. how the insured optimally splits the present value of her contributions over time) of the insured basically can be explained in terms of sticking as close as possible to the Merton solution. For example, in the special case that the insurance company already invests along the lines of the Merton solution, the optimal contribution without an embedded guarantee is given by an upfront contribution of 100% of the present premium value. The same is true for any investment fraction of the insurance company which is lower than the Merton solution (based on the risk aversion of the insured). However, if the investment fraction is higher than the Merton solution, the insured benefits from splitting her contributions over time. The benchmark setup is used to shed some first light on the impact of periodic contributions by analyzing the expected utilities (implied by different premium frequencies) and by comparing them to the Merton solution, in particular in terms of the suboptimality described by the certainty equivalent (loss rate). Further, we overlay the savings plan by some guarantee, i.e. we also account for the impact of the premium payment schemes on guarantee products. Including a guarantee implies that the optimal contribution scheme is implied by an upfront contribution which is given by 100% of the present premium value reduced by the guarantee costs. Again, if the investment strategy of the insurance company deviates from the Merton solution, the optimal contribution scheme of the insured is given by sticking as close as possible to the Merton solution while accounting of the guarantee costs.

However, the interesting part of our contributions concern the introduction of a management rule which defines the investment strategy of the insurance company. As cornerstone examples, we consider three benchmark management rules. The first one prescribes that the riskiness of the investments is reduced in times of falling markets (portfolio insurance strategy). The second one is a constant risk structure. And the third one prescribes to increase the riskiness in times of falling markets (gambling for resurrection). Since the risk structure of the asset side can now depend on the previous market movements, the optimal contribution scheme can not be immediately explained by the Merton solution. We shed light on partly opposing effects. Obviously, a management rule which is lined to previous market movements implies that future deviations from the Merton solution are random. Basically, the risk averse investor neither likes the deviations nor their randomness. In addition, assuming an CRRA investor, she basically dislikes an embedded guarantee. We shed light on the combined effects of random deviations from the Merton solution and the guarantee.

In particular, we show that the directional effects of the combined effects on the optimal contribution scheme can only unambiguously explained in the special case that the initial (and expected future) investment fraction of the insurance company coincides with the Merton solution. Here, an upper bound of the utility of the insured is implied by the second management rule which does not incorporate random devi-
ations from the (overall optimal) Merton solution, i.e. with and without embedded guarantee.

Our paper is related to several strands of the literature including the ones on (i) periodic premium contribution schemes, (ii) pricing and hedging embedded guarantees/options, (iii) utility losses caused by guarantees and/or suboptimal investment decisions (conducted by insurance companies or pension funds), (iv) portfolio planning and (v) innovative guarantee contracts. (vi) regime switching. Without postulating completeness we only refer to a subset of the related literature and hint at the additional literature given within the mentioned papers.

Classic life insurance products contain long term guarantees. The value and risk management of this guarantees have been discussed in the academic literature since the pioneer work of Brennan and Schwartz (1976), where option pricing theory is used for fair pricing of the contracts. Further analysis of this topic can be found in Aase and Persson (1994) and Ekern and Persson (1996). The insurance industry has ignored the pricing problem almost completely for a long time until the last years. Recent academic contributions include Nielsen et al. (2011), Kling et al. (2011), Schmeiser and Wagner (2011), Goecke (2013), Bauer et al. (2008) and Bacinello et al. (2011). For the special case of option pricing in the context of pension funds, we refer to Broeders and Chen (2013).

The first consideration of periodic premium contributions also dates back to Brennan and Schwartz (1976). Referring to periodic premium contribution schemes there have been many other works, e.g. Boyle and Schwartz (1977) who also include mortality risk. Contributions including guarantees payable upon death, survival and surrender are given by Nielsen and Sandmann (1995), Kurz (1996), Nielsen and Sandmann (1996), Nielsen and Sandmann (2002), Bacinello (2003), Gatzert and Schmeiser (2008), Bacinello et al. (2009), Costabile et al. (2009) and Calidonio-Aguilar and Xu (2011). Upper and lower bounds for fair premia are provided by Hürlimann (2010). A further work is Brennan and Solanki (1981) who analyze an optimal investment portfolio insurance contract where the investor spends her whole wealth in one contract by considering single and periodic premia. Further discussions about single and periodic premia can be found in Delbaen (1986), Bacinello and Ortu (1993a), Bacinello and Ortu (1993b) and Bacinello and Ortu (1994).

There is also a huge amount of literature which analyzes the concept of guarantee schemes: To assess life insurance contracts with guarantees and participation on the surplus of the insurer, it is necessary to evaluate the asset strategy itself. Here exists a strong connection to the literature of portfolio optimization. A reference is made to Huang et al. (2008), Milevsky and Kyrychenko (2008), Boyle and Tian (2008), Branger et al. (2010), Gatzert (2013) and Mahayni and Schneider (2016). The relevant literature of portfolio optimization already dates back to Merton (1971) where, among other results, the problem of maximizing the expected utility of an investor
with constant relevant risk aversion (CRRA) is solved in a Black Scholes model setup. The utility maximizing strategy is given in terms of a constant investment fraction in the single assets. This implies that the optimal portfolio is adjusted in continuous time. In contrast, periodic premium contributions imply only the possibility of discrete time adjustments of the portfolio of the insured which, in the optimum, causes a utility loss compared to the optimal continuous time version. Rogers (2001) compares the utility losses of a relaxed investor, who updates her portfolio in discrete time to the classic dynamic Merton problem.\footnote{Rogers shows that the costs of the relaxed investor are small when the discretization of the time lag is smaller than two years, i.e. the portfolio will leave unchanged for $t < 2$ years.}

Accounting for default risk in life insurance products with terminal guarantees were firstly examined by Briys and De Varenne (1997) and Grosen and Jørgensen (2002). More recent papers are Schmeiser and Wagner (2015), Hieber et al. (2016) and Mahayni et al. (2019), where also utility losses caused by suboptimal investment strategies for the insured are analyzed. Other papers with this topic are e.g. Jensen and Sørensen (2001) and Jensen and Nielsen (2016). Surplus participation of the insured including the calculation of her expected utility with focus on the German-speaking countries are analyzed in Maurer et al. (2013).

We further consider a management rule, s.t. the insurer adjusts the investment strategy if positive or negative shocks on the stock market occur. Such management rules may be for example conducted by a regulatory framework and have a regime switching character. An overview of regime switching contributions is given by Elliott et al. (2010). We additionally refer to previous works on regime switching such as Zhou and Yin (2003), Frauendorfer et al. (2007), Costa and Araujo (2008), Xie (2009), Fu et al. (2014) and Capponi (2014).

The further outline of the paper is as follows. In Sec. 2 we present our baseline model assumptions and define the management rule function. Section 3 deals with the savings plan without a management rule. We analyze the optimal premium splitting factor in terms of the expected utility of the insured by considering both, a terminal guarantee and a guarantee which is linked to when the contributions are due. All results are benchmarked to the the special case without a guarantee. In the last section, we analyze the impact of different management rules which can cause a regime switch in the risk profile of the asset side. Again, a terminal guarantee and the special case without a guarantee is considered.

## 2 Baseline model assumptions

We start the analysis within a stylized example which allows to shed some first light on the impact of the investment strategy of the insurance company and the periodic...
premium scheme on the utility of the insured.

Our financial market model over the filtrated probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) is given by the Black and Scholes model. The filtration \((\mathcal{F}_t)_{t \in [0,T]}\) is generated by the standard Brownian motion \((W_t)_{t \in [0,T]}\). Because of the completeness of the Black Scholes model, there exists a uniquely determined equivalent martingale measure \(\mathbb{P}^*\) under which the process \((W^*_t)_{t \in [0,T]}\) defines a standard Brownian motion. In particular, the asset \((S_t)_{t \in [0,T]}\) and bond dynamics \((B_t)_{t \in [0,T]}\) are given by

\[
\begin{align*}
    dS_t &= S_t (\mu dt + \sigma dW_t) = S_t (r dt + \sigma dW^*_t), \quad S_0 = s \\
    dB_t &= B_t r dt, \quad B_0 = b.
\end{align*}
\]

Under the real world probability measure \(\mathbb{P}\), the asset price follows a geometric Brownian motion with constant drift \(\mu (\mu > r)\) and constant volatility \(\sigma (\sigma > 0)\). Under the uniquely defined equivalent martingale measure (pricing measure) \(\mathbb{P}^*\), the asset price follows a geometric Brownian motion with constant drift \(r\) and constant volatility \(\sigma (\sigma > 0)\). The risk-free bond \(B\) grows at a constant interest rate \(r\). The time of maturity is given by \(T < \infty\). Throughout the following we consider a stylized example concerning an investment horizon of \(T = 2\) years.

Now the investor can decide at every point in time \(t\) which proportion of her wealth \(\pi_t (\pi_t \in [0;1])\) is invested in the risky asset \(S\). The remaining part \((1 - \pi_t)\) will be placed in the risk-free bond \(B\). So the evolution of the portfolio wealth, denoted by the stochastic process \((A_t)_{t \in [0,T]}\), is described by

\[
dA_t = A_t \left( \pi_t \frac{dS_t}{S_t} + (1 - \pi_t) \frac{dB_t}{B_t} \right), \quad A_0 = 1. \tag{1}
\]

The drift and volatility sequence \((\mu_{A,t})_{t \in [0,T]}\) resp. \((\sigma_{A,t})_{t \in [0,T]}\) of the value process for every \(t \in [0,T]\) is given by

\[
\begin{align*}
    \mu_{A,t} &:= \pi_t \mu + (1 - \pi_t) r \\
    \sigma_{A,t} &:= \pi_t \sigma.
\end{align*}
\]

For the special case where the investment fraction is constant over time, i.e. \(\pi_t = \pi\), for all \(t \in [0,T]\), we follow a so called Constant Mix strategy (CM). Here the drift and volatility is also constant over time \((\mu_{A,t} = \mu_A; \sigma_{A,t} = \sigma_A, \text{ for all } t \in [0,T])\). In consequence for the solution of equation (1) following a CM strategy it holds under the real world measure \(\mathbb{P}\) and pricing measure \(\mathbb{P}^*\):

\[
A_t = A_0 e^{(\mu_A - \frac{1}{2} \sigma_A^2) t + \sigma_A W_t} = A_0 e^{(r - \frac{1}{2} \sigma^2) t + \sigma_A^* W^*_t}.
\]

In the whole paper we distinguish between two premium dates, where the premium fraction can be payed: \(t = 0\) and \(t = 1\). So it is interesting to analyze the fractions of wealth \(\pi_0\) and \(\pi_1\) in more details. While \(\pi_0\) is prescribed at inception, we allow \(\pi_1\) to depend on the evolution of the stock \(S\) resp. on the Brownian Motion \(W^*\):

\(^2\) For a discussion of different strategies for \(\pi_t\) see for example Balder et al. (2009)
Definition 1
The fraction of wealth $\pi_1$, invested in the risky asset $S$, is defined by

$$\pi_1 := g(W_1 - W_0) = g(W_1),$$

where $g : \mathbb{R} \rightarrow [0, 1]$ is a non-negative function.

Remark 1
(1) Notice the relation between $S_1$ and $W_1$ is given by

$$\frac{S_1}{S_0} = e^{\mu - \frac{1}{2} \sigma^2 + \sigma W_1} \text{ i.e. } W_1 = \frac{1}{\sigma} \left( \ln \left( \frac{S_1}{S_0} \right) - \mu + \frac{1}{2} \sigma^2 \right),$$

so the function $g(W_1)$ can be seen as a transformation of a function $f(\cdot)$, depending on the stock return $\frac{S_1}{S_0}$.

(2) With Definition 1 we are able to model a management rule with regard to the evolution of the stock $S$ resp. of the Brownian Motion $W$. By doing so, we create a link to the regime switching literature.
Positive or negative shocks on the stock market may occur and so the insurer should be able to adjust the investment strategy. Using Definition 1 we can distinguish in a basic example the following three cases for a management rule function $g$:

(i) $g(W_1) = d\pi_0 1_{\{W_1 \leq 0\}} + u\pi_0 1_{\{W_1 > 0\}}$
(ii) $g(W_1) = \pi_0$
(iii) $g(W_1) = u\pi_0 1_{\{W_1 \leq 0\}} + d\pi_0 1_{\{W_1 > 0\}},$

where $u > 1$ and $d < 1$ are non-negative proportional factors. So the choice of $\pi_1$ for the different cases implies the following:

(i) **Same directions**: The insurance company increases the riskiness of the investments in times of increasing markets

(ii) **Constant**: The fraction is constant

(iii) **Opposite directions**: The riskiness reduces in times of increasing markets

The preferences of the insured are given in terms of a constant level relative risk aversion (CRRA) denoted by $\gamma$, i.e. a utility function of the form

$$u^{(CRRA)}(x) := \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \text{for } \gamma > 1 \\ \ln(x), & \text{for } \gamma = 1. \end{cases}$$
The use of CRRA utility allows us an analysis only based on the asset returns in later sections. Thereby we determine the expected utility of the insured. We distinguish between an upfront premium and periodic premia in order to examine which one leads to a higher expected utility. This is combined with a guarantee feature for the insured (Put-option) and a management rule on the side of the insurer. Results are stated in closed form and via a simulation study.

With this analysis we are able to assess if (and in which constellation) periodic premia generate benefits for the insured based on different levels of constant relative risk aversion \( \gamma \) and the expected utility for the insured in terms of the certainty equivalent \( CE \).

3 Savings plan without management rule

The insured participates with her saving portion in the asset side of the insurance company. The existing literature focuses on participating life insurance contracts where an upfront premium is implicitly assumed. In this paper, we analyze different contribution schemes for a maturity \( T = 2 \), also including a management rule. We take into account the standard upfront premium and additionally a postponed premium where the whole contribution is paid in \( t = 1 \). The main focus in this section is on the combination of this two settings: We allow the insured to pay her premia periodically, i.e. she pays one part of her contribution in \( t = 0 \) and the remaining part in \( t = 1 \). We further assume that these contributions are normalized to 1. Thereby it is prescribed by regulation that the insured receives at least a guaranteed value to reduce the a possible loss. Due to that, we include a guarantee component as well.

Here we focus on a constant investment fraction \( \pi \) over time, i.e. \( \pi_0 = \pi_1 = \pi \). This is in line with the constant case of the management rule function \( g(W_1) \). In the next section we discuss the different cases of the function and its implications on the expected utility of the insured in more details.

At the first premium date \( t = 0 \), the insured contributes the amount \( \beta \) \((0 \leq \beta \leq 1)\) and the second one \((1 - \beta)e^r\) at \( t = 1 \). The special case \( \beta = 1 \) \((\beta = 0)\) refers to a single upfront (postponed) premium. Thus, the contribution scheme is given in terms of the so called premium fraction \( \beta \). Notice that at \( t = 0 \), \( \beta \) is invested in the asset side \( A \) of the insurer (i.e. \( \beta \pi \) in \( S \)) and the remaining part \((1 - \beta)\) is invested in a risk free bond with interest rate \( r \). After \( t = 1 \), the insured is fully invested in the asset side \( A \) of the insurer. Consequently, she has no more influence on the riskiness of her investment and her investment fraction (in \( S \)) is equal the investment fraction of the insurance company \( \pi \).

We introduce a terminal guarantee \( G_2 \) such that the investor receives, at time \( T = 2 \),
the higher value of her account value $V_2$ and a guaranteed value $G_2$, i.e. the payoff to the insured is given by $\max\{V_2, G_2\}$. We assume that $G_2$ is given by a continuously compounded risk free interest rate $g$, s.t. $G_2 = e^{2g}$. We further take into account an adapted guarantee where the premium fractions have an additional impact the guarantee value. Furthermore, when $g \to -\infty$ there exists no guarantee, i.e $G_2 = 0$.

Obviously, the guarantee is not for free. For the sake of simplicity, we assume that the insured pays an upfront fee for the guarantee. Thus, the expected utility of different payoffs is comparable if the upfront fees (and the present value of the contributions) coincide.

In the first instance, we analyze the dependence of the value of a fix terminal guarantee on the premium fraction $\beta$. By no arbitrage arguments, the value of the guarantee (insurance put option with payoff $[G_2 - V_2]^+$) is given by the expected discounted payoff under the pricing measure $P^*$, i.e.

$$Put_0 = e^{-2r} E_{P^*} \left[ (G_2 - V_2)^+ \right].$$

Here, the terminal wealth of the insurance company $V_2$ is given by

$$V_2 = \beta \frac{A_2}{A_0} + (1 - \beta) e^r \frac{A_2}{A_1}$$

$$= \beta \frac{A_1 A_2}{A_0 A_1} + (1 - \beta) e^r \frac{A_2}{A_1},$$

where

$$\frac{A_{i+1}}{A_i} = e^{\mu_A - \frac{1}{2} \sigma_A^2 + \sigma_A (W_{i+1} - W_i)},$$

$$W_{i+1} - W_i \sim N(0, 1)$$

and

$$\mu_A = \pi \mu + (1 - \pi) r = r + \pi (\mu - r)$$

and $\sigma_A = \pi \sigma$.

In the special case $\beta = 1$, it holds

$$Put_0^{(\beta=1)} = e^{-2r} E_{P^*} \left[ (G_2 - \frac{A_2}{A_0})^+ \right],$$

such that the put price is given by the BS price of a European put option with underlying $A$ (where $A_0 = 1$), with cumulated volatility $\sigma_A T = 2\pi \sigma$ and strike $K = G_T$.

For $\beta = 0$, it holds (since $\frac{A_2}{A_1}$ and $\frac{A_1}{A_0}$ are equal in distribution)

$$Put_0^{(\beta=0)} = e^{-2r} E_{P^*} \left[ (G_2 - e^r \frac{A_2}{A_1})^+ \right] = e^{-r} E_{P^*} \left[ (e^{-r} G_2 - \frac{A_2}{A_1})^+ \right]$$

$$= e^{-r} E_{P^*} \left[ (e^{-r} G_2 - A_1 \frac{A_0}{A_1})^+ \right]$$.
Put price for different guarantees and different investment fractions

Figure 1: The parameters we used are $r = 0.01$, $\sigma = 0.2$ and $T = 2$. Additionally, for the left figure we set $\pi = 0.35$ and for the right figure $g = 0.009$. The black line refers to $\beta = 0$, the black dashed line to $\beta = 1$ and the gray dashed one to $\beta = 0.5$.

such that the put price is given by the BS price of a European put option (with maturity $T = 1$) with underlying $A$ (where $A_0 = 1$), cumulated volatility $\sigma_A = \pi \sigma$ and strike $K = e^{-rG_2}$.

The price of the guarantee depends on different parameters such as the promised guarantee rate $g$ and the investment fraction $\pi$.

The impact of the promised guarantee rate $g$ on the put price is shown in the left-hand side of figure 1. We distinguish between different premium fractions $\beta$. We receive the following results:

The put price is strictly monotonically increasing in the guarantee rate $g$, which is intuitively clear. Another obvious result is that the put price is lower for a postponed premium payment ($\beta = 0$) than for an initial premium payment ($\beta = 1$): For $\beta = 1$ the insurance period is longer (and therefore more expensive) because the insured already invests at $t = 0$ whereas for $\beta = 0$ she invests later ($t = 1$). Furthermore it is interesting to see that the price for $\beta = 0.5$ is closer to the one of $\beta = 0$ than to $\beta = 1$.

The impact of the investment fraction $\pi$ on the put price is shown in the right-hand side of figure 1. Again, we distinguish between different premium fractions $\beta$. The put price is strictly monotonically increasing in the investment fraction $\pi$, which is intuitively clear because a higher investment fraction leads to a riskier portfolio. The relation of the put prices for different $\beta$ is the same as in the guarantee case in the left-hand side figure.

Additionally to the put price we can analyze the terminal wealth of the insured
(liability side) given by

\[ L_2 := V_2 + (G_2 - V_2)^+ = \max\{V_2, G_2\}. \]

We further assume that the put price (the guarantee) has to be financed by the wealth of the insured (normalized to 1). Therefore the sum of the premium fractions has to be smaller than 1, because the remaining part refers to the put price.\(^3\) Consequently, the notation has to be adjusted, such that the premium fraction at \( t = 0 \) is given by \( \beta_0 \) and the second premium fraction at \( t = 1 \) is denoted by \( \beta_1 \). The wealth of the insured (normalized to 1) at \( t = 0 \) is then split as follows:

\[ \beta_0 + \beta_1 + \text{Put}_0 = 1. \]

Additionally to this, the terminal wealth of the insured is now defined as

\[ L_2 := \tilde{V}_2 + (G_2 - \tilde{V}_2)^+, \]

where

\[ \tilde{V}_2 := \beta_0 \frac{A_2}{A_0} + \beta_1 e^{r \frac{A_2}{A_1}}. \]

The fair pricing of the contract is shown in the following Lemma.

**Lemma 1**

The fair pricing of the insured’s terminal wealth \( L_2 \) is given by

\[ e^{-2r} \mathbb{E}_P^*[L_2] = \beta_0 + \beta_1 + \text{Put}_0 = 1. \]

For the proof of Lemma 1 see Appendix A.

Throughout the following, let \( P_t^{BS}(S_t, T - t, K, \sigma) \) denotes the \( t \)-price of a European put option with underlying \( S \), current asset price \( S_t \), time to maturity \( T - t \) in a Black and Scholes model setup where the dynamics of \( S \) are given in terms of a geometric Brownian motion with volatility \( \sigma \) s.t.

\[ P_t^{BS}(S_t, T - t, K, \sigma) = e^{-r(T-t)} K \Phi(-d_2(S_t, T - t, K, \sigma)) \]

\[ - S_t \Phi(-d_1(S_t, T - t, K, \sigma)), \]

where

\[ d_1(S_t, T - t, K, \sigma) := \frac{\ln \frac{S_t}{K} + (\frac{1}{2} \sigma^2 + r)(T - t)}{\sigma \sqrt{T - t}} \]

and

\[ d_2(S_t, T - t, K, \sigma) := d_1(S_t, T - t, K, \sigma) - \sqrt{T - t} \sigma. \]

\( \Phi(\cdot) \) denotes the distribution function of the standard normal distribution.

For the special case \( \beta_1 = 0 \) and maturity \( T = 2 \) the maximum premium fraction

\(^3\) Notice, when the put price is equal to 0, we refer to the savings plan without guarantee (Section 2.1)
at inception is given by \( \beta_0^{Max} = 1 - Put_0 \). Notice that the put price here can be calculated in closed form, i.e.

\[
Put_0 = e^{-2rE_p} \left[ \left( G_2 - \beta_0 \frac{A_2}{A_0} \right)^+ \right] \\
= \beta_0 e^{-2rE_p} \left[ \left( \frac{G_2}{\beta_0} - \frac{A_2}{A_0} \right)^+ \right] \\
= \beta_0 P_0^{BS} \left( 1, 2, K = \frac{G_2}{\beta_0}, \pi \sigma \right).
\]

(2)

In formula (2) we can see, that the put price \( Put_0 \) is increasing in the investment fraction \( \pi \). This result is intuitively clear because a higher investment fraction leads to a riskier portfolio and so the guarantee is more pricy.

Because both sides of the equation depend on \( \beta_0 \), the maximized premium fraction \( \beta_0^{Max} \) has to be determined by solving a fix-point problem. By setting \( \beta_0 = 0 \), the maximized premium fraction \( \beta_1^{Max} \) can be determined in the same way. A closer analysis of this is given in the next section.

We are interested in the expected utility of the insured, i.e. in \( E_p[u(L_2)] \). We are able to determine the expected utility in closed form for the two special cases \( \beta_0 = 0 \) (implying \( \beta_1 = \beta_1^{Max} \)) and \( \beta_1 = 0 \) (implying \( \beta_0 = \beta_0^{Max} \)). With this result we can also easily calculate the corresponding certainty equivalents \( CE \), implicitly defined by the equation \( u(CE) = E_p[u(L_2)] \) and the savings rates \( y(\pi) := \frac{1}{2} \ln(CE) \) in closed form.

**Proposition 1**

Let \( L_2 \) be the terminal wealth of the insured, \( u(\cdot) \) a CRRA utility function, \( G_2 = e^{2g} \) the terminal guarantee feature and \( \Phi(\cdot) \) the distribution function of the standard normal distribution.

(a) For \( \beta_1 = 0 \) the following results hold:

\[
(i) \quad E_p[u(L_2^{(\beta_1=0)})] = \frac{(\beta_0^{Max})^{(1-\gamma)}}{1-\gamma} e^{(1-\gamma)(2\mu_A-\gamma\sigma_A^2)} \left\{ 1 - \Phi \left( \frac{\ln\left( \frac{G_2}{\beta_0^{Max}} \right) - 2[\mu_A - \sigma_A^2(\gamma - \frac{1}{2})]}{\sqrt{2}\sigma_A} \right) \right\} \\
+ \frac{1}{1-\gamma} G_2^{(1-\gamma)} \Phi \left( \frac{\ln\left( \frac{G_2}{\beta_0^{Max}} \right) - 2[\mu_A + \sigma_A^2]}{\sqrt{2}\sigma_A} \right).
\]

\[
(ii) \quad CE_2^{(\beta_1=0)} = \left\{ (\beta_0^{Max})^{(1-\gamma)} e^{(1-\gamma)(2\mu_A-\gamma\sigma_A^2)} \left\{ 1 - \Phi \left( \frac{\ln\left( \frac{G_2}{\beta_0^{Max}} \right) - 2[\mu_A - \sigma_A^2(\gamma - \frac{1}{2})]}{\sqrt{2}\sigma_A} \right) \right\} \\
\right.
\]
\[ + G_2^{(1-\gamma)} \Phi \left( \frac{\ln(\frac{G_2}{\beta_{Max}}) - 2\mu_A + \frac{1}{2} \gamma \sigma_A^2}{\sqrt{2}\sigma_A} \right) \right) \frac{1}{1-\gamma}.

(b) For \( \beta_0 = 0 \) the following results hold:

(i) \[ \mathbb{E}[u(L_2^{(\beta_0=0)})] = \frac{(\beta_{Max})^{(1-\gamma)} e^{(1-\gamma)(r+\mu_A - \frac{1}{2} \gamma \sigma_A^2)}}{1-\gamma} \left\{ 1 - \Phi \left( \frac{\ln(\frac{G_2}{\beta_{Max}}) - r - \mu_A - \frac{1}{2} \gamma \sigma_A^2}{\sigma_A} \right) \right\} + \frac{1}{1-\gamma} G_2^{(1-\gamma)} \Phi \left( \frac{\ln(\frac{G_2}{\beta_{Max}}) - r - \mu_A}{\sigma_A} \right) \right). \]

(ii) \[ CE_2^{(\beta_0=0)} = \left\{ (\beta_{Max})^{(1-\gamma)} e^{(1-\gamma)(r+\mu_A - \frac{1}{2} \gamma \sigma_A^2)} \right\} \left\{ 1 - \Phi \left( \frac{\ln(\frac{G_2}{\beta_{Max}}) - r - \mu_A - \frac{1}{2} \gamma \sigma_A^2}{\sigma_A} \right) \right\} + G_2^{(1-\gamma)} \Phi \left( \frac{\ln(\frac{G_2}{\beta_{Max}}) - r - \mu_A}{\sigma_A} \right) \right)^{\frac{1}{1-\gamma}}. \]

For the proof of Proposition 1 see Appendix B.

For \( g \to -\infty \), which refers to a no guarantee case, corollary 1 follows immediately:

**Corollary 1**

We assume the same assumptions as in Proposition 1. For the special case \( g \to -\infty \) it holds that \( L_2 = V_2 \), s.t. the following results hold.

(a) Upfront premium \( \beta = 1 \):

(i) \[ \mathbb{E}[u(V_2^{(\beta=1)})] = \frac{1}{1-\gamma} e^{2(1-\gamma)(\mu_A - \frac{1}{2} \gamma \sigma_A^2)} \]

(ii) \[ CE_2^{(\beta=1)} = e^{2(\mu_A - \frac{1}{2} \gamma \sigma_A^2)} \]

(iii) \[ y^{(\beta=1)}(\pi) = \mu_A - \frac{1}{2} \gamma \sigma_A^2 \]

(iv) \[ l^{(\beta=1)}(\pi) = \frac{1}{2} \gamma \sigma^2 (\pi^* - \pi)^2 \]

(b) Postponed premium \( \beta = 0 \):

(i) \[ \mathbb{E}[u(V_2^{(\beta=0)})] = \frac{1}{1-\gamma} e^{(1-\gamma)(r+\mu_A - \frac{1}{2} \gamma \sigma_A^2)} \]

(ii) \[ CE_2^{(\beta=0)} = e^{(r+\mu_A - \frac{1}{2} \gamma \sigma_A^2)} \]

(iii) \[ y^{(\beta=0)}(\pi) = \frac{1}{2} (r + \mu_A - \frac{1}{2} \gamma \sigma_A^2) \]

(iv) \[ l^{(\beta=0)}(\pi) = \frac{1}{2} l^{(\beta=1)}(\pi) + \frac{1}{4} \gamma \sigma^2 (\pi^*)^2 \]

12
Remark 2
The solution of Proposition 1 can easily be generalized for maturity $T \neq 2$. The savings rates and loss rates in Corollary 1 are easily calculated.

The optimization problem can be stated as follows (without formulating the optimization arguments):

$$\max \mathbb{E}_{\mathcal{F}}[u(L_2)] \text{ s.t. } e^{-2r\mathbb{E}_{\mathcal{F}}[L_2]} = 1.$$  \hfill (3)

The strategy of the insurance company is prescribed, s.t. the insured is only able to influence her expected utility by choosing the amount of the premium fractions $\beta_i$. In combination with Lemma 1, we can ensure that the constraint (3) is fulfilled if $\beta_0 + \beta_1 + Put_0 = 1$. So the above optimization problem can be written in the following way:

$$\max_{\{\beta_0, \beta_1 \in [0, 1]; \beta_0 + \beta_1 + Put_0 = 1\}} \mathbb{E}_{\mathcal{F}}[u(L_2)].$$ \hfill (4)

Because we can not solve this problem in closed form, we use simulations. Therefore, we numerically calculate the expected utility for suitable $(\beta_0, \beta_1)$-tupels with $(\beta_0, \beta_1) \in \mathcal{C} = \{(0, \beta_1^{Max}), \ldots, (\beta_0^{Max}, 0)\}$, where $\beta_0$ increases with $\Delta \beta_0 = 0.05$ and find\footnote{We choose $\Delta \beta_0 = 0.05$ due to speed of calculation.}

$$(\beta_0^*, \beta_1^*) := \arg\max_{\beta_0, \beta_1 \in \mathcal{C}} \mathbb{E}[u(V_2 + (G_2 - V_2)^+)].$$

We now want to determine the optimal premium fractions, including a terminal guarantee $G_T = e^{gT}$ with $g \in \{-\infty, -0.005, 0, 0.009\}$. Notice that $g = 0$ refers to the preservation of capital and not to a savings plan without guarantee.

First, we analyze the no guarantee case $g \to -\infty$ which is a special case of optimization problem (4). Here, the put price is 0 and therefore the premium fractions sum up to 1. So the constraint of the fair pricing condition is always fulfilled, s.t. the optimization problem is given by

$$\max \mathbb{E}[u(V_2)].$$

As known from the famous paper of Merton (1971) the overall expected utility maximizing strategy is given by a CM-strategy with the constant fraction of wealth $\pi^* = \pi^{Mer} := \frac{\mu - r}{\sigma^2}$. For the special case, that the investment fraction $\pi$ is equal to...
\(\pi_{Mer}\), the optimal expected utility is archived by setting \(\beta = 1\) (upfront premium). For the case that the investment fraction differs from the Merton fraction \(\pi_{Mer}\), it is interesting to analyze how the optimal premium fractions change. A first hint is given by the following Proposition which compares the savings rate for upfront and postponed premium contributions.

**Proposition 2**

Let \(y^{(\beta=0)}(\pi)\) and \(y^{(\beta=1)}(\pi)\) be the savings rates as defined above. Then it holds

\[
y^{(\beta=0)}(\pi) > y^{(\beta=1)}(\pi) \iff \pi > 2\frac{\mu - r}{\gamma \sigma^2} \iff \pi > 2\pi^*.
\]

The proposition shows, that postponing (completely) the premium to \(t = 1\) has its merits for the insured iff the insurers investment fraction in the risky asset is twice the optimal individual Merton fraction. On the other hand, an upfront contribution is beneficial iff the investment fraction of the insurer is lower than twice the Merton fraction.

However, the insured is also able to split her premium fraction \(\beta\) in order to invest one part at \(t = 0\) and the remaining part at \(t = 1\), s.t. \(0 < \beta < 1\). In the following we discuss the utility changes of the insured and determine her optimal expected utility in this setting. For a given investment fraction, the optimal (expected utility maximizing) premium fraction \(\beta^*\) where

\[
\beta^* := \arg\max_{\beta \in [0,1]} \mathbb{E}[u(V)] = \arg\max_{\beta \in [0,1]} \mathbb{E}\left[u\left(\frac{\beta A_2}{A_0} + (1 - \beta)e^{r\frac{A_2}{A_1}}\right)\right]
\]

(5)

can only be determined numerically, if the strategy \(\pi\) which is implemented by the insurer does not coincide with the Merton solution (implying \(\beta^* = 1\)).

Intuitively, it is clear that the optimal premium fraction \(\beta^*\) is determined independently of \(\frac{A_2}{A_1}\). Determine \(\beta\) at inception implicitly sets the remaining fraction \((1 - \beta)\) which is payed at \(t = 1\). So the investor is 100% invested in the asset side of the insurer for \(t \in [1, 2]\). This intuition can also be seen mathematically: Notice that the
first order condition of the above optimization problem is of the following form:

\[
\mathbb{E} \left[ u'(V_2) \left( \frac{A_1 A_2}{A_0 A_1} - e^{r \frac{A_2}{A_1}} \right) \right] = 0
\]

\[
\Leftrightarrow \mathbb{E} \left[ \left( \frac{A_1 A_2}{A_0 A_1} + (1 - \beta) e^{r \frac{A_2}{A_1}} \right)^{-\gamma} \left( \frac{A_1 A_2}{A_0 A_1} - e^{r \frac{A_2}{A_1}} \right) \right] = 0
\]

\[
\Leftrightarrow \mathbb{E} \left[ \left( \frac{\beta A_1 A_0}{A_0 A_1} + (1 - \beta) e^{r} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^{r} \right) \left( \frac{A_2}{A_1} \right)^{1-\gamma} \right] = 0. \tag{6}
\]

Since \( \frac{A_1}{A_0} \) and \( \frac{A_2}{A_1} \) are independent, the FOC thus simplifies to

\[
\mathbb{E} \left[ \left( \frac{\beta A_1 A_0}{A_0 A_1} + (1 - \beta) e^{r} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^{r} \right) \right] = 0.
\]

More detailed calculations concerning the optimal premium fraction can be found in the next section.

Apart from the special case, we consider the guarantee rates \((g \in \{-0.005, 0, 0.009\})\) as mentioned before. As in the special case the overall optimal solution for the expected utility is well known. The solution dates back to Basak and Shapiro (2001). An application can be found for example in Mahayni et al. (2019) where the 'optimal quantile payoff' is given by

\[
\beta \frac{A_{Mer}^2}{A_0} + \left( G_2 - \beta \frac{A_{Mer}^2}{A_0} \right)^+, \text{ where}
\]

\[
1 - \beta = \mathbb{E}_P \left[ e^{-2r} \left( G_2 - \beta \frac{A_{Mer}^2}{A_0} \right)^+ \right].
\]

\( A_{Mer}^2 \) refers to the strategy where the investment fraction is given by the Merton fraction. This finding shows that for \( \pi = \pi^{Mer} \) the postponed premium fraction is set to zero \( \beta_1 = 0 \) s.t. the upfront premium fraction is maximized (Notice, that the put here is not for free, i.e. \( \beta_0 \neq 1 \)). Again, it is interesting to analyze the impact of \( \pi \) on the optimal premium fraction, when the investment fraction differs from the Merton solution.

Therefore, numerical illustrations are based on a Black and Scholes parameter setup (for a risky asset \( S \) and a zero bond \( B \)) with

\[
r = 0.01; \ \mu = 0.05; \text{ and } \sigma = 0.2.
\]

5 Here, changing the derivation and expectation is possible because of the dominated convergence theorem: \( \frac{\partial}{\partial \beta} \mathbb{E} \left[ u \left( \beta \frac{A_2}{A_0} + (1 - \beta) e^{r \frac{A_2}{A_1}} \right) \right] = \mathbb{E} \left[ \frac{\partial}{\partial \beta} u \left( \beta \frac{A_2}{A_0} + (1 - \beta) e^{r \frac{A_2}{A_1}} \right) \right]. \)
Due to the guarantee costs, we have to consider the optimal adjusted premium fractions 
\[ \tilde{\beta}_0^* := \frac{\beta_0^*}{(\beta_0^* + \beta_1^*)} \text{ and } \tilde{\beta}_1^* := \frac{\beta_1^*}{(\beta_0^* + \beta_1^*)} = 1 - \tilde{\beta}_0^*. \]

An adjustment of the optimal premium fractions is necessary, because of the reduced present value of the insured\'s contribution. By doing so, we can now compare these ratios with the Merton solution where the upfront premium is now given by \( \beta_0^* + \beta_1^* \).

As mentioned above, we maximize the expected utility of the insured or equivalently minimize her loss rate. The results for a risk aversion \( \gamma = 4 \) are stated in Table 1. We distinguish between four guarantee rates: \( g \in \{ g \to -\infty, -0.005, 0, 0.009 \} \).

In the first column the different investment fractions are stated. The second column refers to the different guarantees and the third column shows the certainty equivalent (loss rate). Column four and five point out the optimal adjusted premium fractions \( \tilde{\beta}_0^* \text{ and } \tilde{\beta}_1^* \), given by the ratio of \( \beta_0^* \text{ (} \beta_1^* \text{)} \) and the total premium fractions. This is due to the adjusted wealth \( 1 - Put_0 \). Notice, that for \( g \to -\infty \) \( \tilde{\beta}_i^* \) is equal to \( \beta_i^* \) because the put price causes no costs and therefore the whole contributions (normalized to 1) are used for the premium fractions. The last column states the total premium fractions. The remaining part up to 1 refers to the put price.

Throughout all guarantee rates it holds that \( \tilde{\beta}_0^* \) and the sum \( \beta_0^* + \beta_1^* \) are monotonically decreasing in the investment fraction. This is intuitively clear because a higher investment fraction induces a riskier portfolio and therefore a higher put price as shown before. Another result is that the put price is increasing in the guarantee rate \( g \) and \( \beta_1^* \) is monotonically increasing.

For \( \pi \leq \pi^{Mer} \) and all guarantee rates \( g \), the optimal adjusted premium fractions \( \tilde{\beta}_1^* \) are 0. The insured invests all of her money at inception of the contract to maximize the expected utility and thus, follows the Merton strategy.

For \( \pi > \pi^{Mer} \) (and all guarantee rates), one can observe that the optimal adjusted premium fraction \( \tilde{\beta}_0^* \) gets smaller to balance the overinvestment in the stock. Remark that at \( t = 0 \), the insured is able to decide about the invested amount \( \beta \pi \) in the stock. The remaining part will be invested at the risk free rate.\(^6\)

\(^6\) However, this is still the case accounting for the reduction of the present value of the contributions because of the up front put premium, i.e. now we have \( \pi_{0(\text{Insured})} = \frac{\beta_0 \pi}{1 - Put_0} \).
Optimal premium fractions $\tilde{\beta}^*_i$ for $\gamma = 4$ and different guarantee rates $g$

<table>
<thead>
<tr>
<th>$\pi$</th>
<th>$g$</th>
<th>$CE^*$</th>
<th>$\tilde{\beta}^<em>_0 = \frac{\tilde{\beta}^</em>_0}{\tilde{\beta}^<em>_0 + \tilde{\beta}^</em>_1}$</th>
<th>$\tilde{\beta}^<em>_1 = \frac{\tilde{\beta}^</em>_1}{\tilde{\beta}^<em>_0 + \tilde{\beta}^</em>_1}$</th>
<th>$\beta^<em>_0 + \beta^</em>_1$</th>
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<td>1.0300</td>
<td>1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
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Table 1: The table presents for $T = 2, \gamma = 4$, and $g \in \{g \rightarrow -\infty, -0.005, 0, 0.009\}$ the optimal certainty equivalents for varying investment fractions by determining the optimal premium fraction $\beta^*_0$. We prescribed a set of possible premium fractions $\beta^*_0 \in \{0, 0.05, \ldots, \beta^*_{\text{Max}}\}$ and choose the one which maximizes the expected utility. Notice that here $\pi^*_{\text{Mer}} = 0.25$ and $CE^*_{\text{Mer}} = 1.0305$.

In a next step we compare the certainty equivalents from the periodic payments with the ones, resulting from a single investment at $t = 0$ (upfront premium) or $t = 1$ (postponed premium) as shown in Table 2. The differences in the certainty equivalents of the insured are crucially for increasing investment fractions $\pi$: For an investment fraction $\pi \leq \pi^*_{\text{Mer}}$ it is clear, that the certainty equivalents of the upfront premium and the periodic ones coincide due to the Merton solution. With increasing investment fractions the differences between the certainty equivalents enlarge. This shows the importance of the periodic premium concept in the context of maximizing the expected utility of the insured. We can also see that $CE^*_{\text{upfront}}$ is always greater than $CE^*_{\text{postponed}}$ with the exception of $g \rightarrow -\infty$. Here, this statement is valid until $\pi < 2\pi^*_{\text{Mer}}$. The relation changes for greater investment fractions, i.e. $CE^*_{\text{postponed}} > CE^*_{\text{upfront}}$. This can be explained with the results of Proposition 2.
An illustration is shown in Figure 2.

**Comparison of optimal certainty equivalents \( CE^* \) for \( \gamma = 4 \)**

<table>
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<tr>
<th>( \pi )</th>
<th>( g )</th>
<th>( CE^* )</th>
<th>( CE_{\text{upfront}} )</th>
<th>( CE_{\text{postponed}} )</th>
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</table>

Table 2: The table presents for \( T = 2, \gamma = 4, \) and \( g \in \{ g \to -\infty, -0.005, 0, 0.009 \} \) the optimal certainty equivalents for varying investment fractions (see Table 1). In addition, we state the certainty equivalents for upfront \((\beta = 1)\) and for postponed \((\beta = 0)\) premia.

Additionally to the former results, we now shed a light on the influence of the premium fractions on the guarantee value. For that we modify the guarantee provided by the insurance company in the following way: We connect the premium fractions to the guarantee, i.e. we do not longer prescribe a fixed guaranteed value. Instead, we prescribe a fixed interest on the premium fractions. The adapted guarantee value (depending on the choice of \( \beta \)) \( \hat{G} \) for maturity \( T = 2 \) is then given by

\[
\hat{G}_2 := \hat{\beta}_0 e^{2g} + \hat{\beta}_1 e^g e^g.
\]  

(7)

So the choice of the premium fractions has an impact on both: the asset value and additionally on the guarantee component. The price for the guarantee is again given by a put option, where
Loss rates for varying investment fractions ($\beta = 1$, $\beta = 0.5$ and $\beta = 0$)

![Graph showing loss rates for varying investment fractions.]

Figure 2: The thick black line refers to $\beta = 1$ (upfront premium), the black dashed line to $\beta = 0.5$ (the present value of the two periodic premia coincide) and the grey dashed one to $\beta = 0$ (the premium is postponed to $t = 1$). Notice that the right hand figure refers to a level of risk aversion $\gamma = 4$ implying a Merton fraction of $\pi^* = 0.25$ (highlighted by the first grid line). The left (right) figure refers to a level of relative risk aversion $\gamma = 2$ ($\gamma = 4$).

As in the previous section, it holds that the price for the guarantee is self-financed, i.e. $\hat{\beta}_0 + \hat{\beta}_1 + \hat{P}ut_0 = 1$.

The terminal wealth of the insured is then given by

$$\hat{L}_2 := \hat{V}_2 + (\hat{G}_2 - \hat{V}_2)^+$$

$$= \left( \hat{\beta}_0 \frac{A_2}{A_0} + \hat{\beta}_1 e^r \frac{A_2}{A_1} \right) + \left( \hat{\beta}_0 e^{2g} + \hat{\beta}_1 e^r e^{2g} - \left[ \hat{\beta}_0 \frac{A_2}{A_0} + \hat{\beta}_1 e^r \frac{A_2}{A_1} \right] \right)^+$$

$$= \left( \hat{\beta}_0 \frac{A_2}{A_0} + \hat{\beta}_1 e^r \frac{A_2}{A_1} \right) + \left( \hat{\beta}_0 \left[ e^{2g} - \frac{A_2}{A_0} \right] + \hat{\beta}_1 e^r \left[ e^{2g} - \frac{A_2}{A_1} \right] \right)^+.$$

The fair pricing of the contract is shown in the following Lemma.

**Lemma 2**

The fair pricing of the insured’s terminal wealth $\hat{L}_2$ is given by

$$e^{-2rE_{\mathbb{P}^*}[\hat{L}_2]} = \hat{\beta}_0 + \hat{\beta}_1 + \hat{P}ut_0 = 1.$$

The proof of Lemma 2 is similar to Lemma 1 which is shown in Appendix A.
Optimal premium fractions $\tilde{\beta}_i^*$ for $\gamma = 4$ and different guarantee rates $g$

<table>
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<tr>
<th>$\pi$</th>
<th>$g$</th>
<th>$CE^*$</th>
<th>$\beta_0^* = \frac{\beta_0}{\beta_0 + \beta_1}$</th>
<th>$\beta_1^* = \frac{\beta_1}{\beta_0 + \beta_1}$</th>
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Table 3: The table presents for $T = 2$, $\gamma = 4$ and $g \in \{-0.005, 0, 0.009\}$ the optimal certainty equivalents for varying investment fractions by determining the optimal premium fraction $\beta_0^*$. We prescribed a set of possible premium fractions $\beta_0^* \in \{0, 0.05, \ldots, \beta_0^{Max}\}$ and choose the one which maximizes the expected utility. Additionally, the adapted guarantee value $\hat{G}_2$ (depending on the choice of $\beta$) is shown in the last column. Notice that here $\pi^{Mer} = 0.25$.

As before, we maximize the expected utility of the insured. The results for a risk aversion $\gamma = 4$ are shown in Table 3. Again, we consider the certainty equivalent $CE$, the optimal adjusted premium fractions $\beta_0^*$ and $\beta_1^*$ and the total premium fractions. Additionally, we take into account the adapted guarantee value $\hat{G}_2$.

The differences in the certainty equivalents are again crucially for increasing investment fractions $\pi$: In contrast to the results from Table 1, the insured invests all of her wealth at inception until $\pi > 0.3$. Obvious, the guarantee value $\hat{G}_2$ is increasing in the guarantee $g$ but decreasing in the investment fraction $\pi$. One can observe that $\beta_0^*$ ($\beta_1^*$) is strictly decreasing (increasing) in $\pi$ for $\pi > 0.3$. Another observation is that the costs for the guarantee (put price) is increasing even though the guarantee value $\hat{G}_2$ is decreasing. The put price gets larger and therefore the remaining wealth of the insured gets smaller. Due to the definition of $\hat{G}_2$ as shown in equation (7), the guarantee value $\hat{G}_2$ depends on the allocation of the premium fractions.
4 Savings plan with management rule

We now discuss the impact of the management rule on the expected utility of the insured and the choice of the optimal premium fractions. While the previous chapter deals with the special case $g(W_1) = \pi$, we now consider a management rule function $g$ to show the impact of the stock evolution on the expected utility of the insured. Therefore, we consider the no guarantee and the guarantee cases as in the previous section. Before starting, we need to adapt our notation.

The terminal wealth of the insurance company $V_2$ is given by

$$V_2 = \beta \frac{A_2}{A_0} + (1 - \beta)e^{r} \frac{A_2}{A_1} = \beta \frac{A_1}{A_0} \frac{A_2}{A_1} + (1 - \beta)e^{r} \frac{A_2}{A_1}$$

where

$$\frac{A_{i+1}}{A_i} = e^{\mu_{A,i} - \frac{1}{2}\sigma_{A,i}^2 + \sigma_{A,i}(W_{i+1} - W_i)},$$

$$W_{i+1} - W_i \sim N(0, 1)$$

and

$$\mu_{A,i} = \pi_i \mu + (1 - \pi_i) r = r + \pi_i (\mu - r)$$

and

$$\sigma_{A,i} = \pi_i \sigma.$$  

As in Definition 1, it holds

$$\pi_0 = \text{const. and } \pi_1 = g(W_1 - W_0) = g(W_1),$$

i.e. the investment fraction is a function of $W_1$ such that, in general, $\frac{A_1}{A_0}$ and $\frac{A_2}{A_1}$ are not independent. Before we start analyzing the expected utility of the insured, we take a step back and have a closer look at the dependence structure of the terminal wealth. By calculating the expected value of $V_2$, it turns out that the covariance between $W_1$ and $g(W_1)$ matters:

$$\mathbb{E}[V_2] = \mathbb{E} \left[ \beta \frac{A_2}{A_0} + (1 - \beta)e^{r} \frac{A_2}{A_1} \right]$$

$$= \beta \mathbb{E} \left[ \frac{A_1}{A_0} \mathbb{E} \left[ \frac{A_2}{A_1} | \mathcal{F}_1 \right] \right] + (1 - \beta)e^{r} \mathbb{E} \left[ \mathbb{E} \left[ \frac{A_2}{A_1} | \mathcal{F}_1 \right] \right]$$

$$= \beta \mathbb{E} \left[ \frac{A_1}{A_0} e^{\mu_{A,1}} \right] + (1 - \beta)e^{r} \mathbb{E} \left[ e^{\mu_{A,1}} \right]$$

$$= e^{2r} \left( \beta e^{\pi_0(\mu - r)} \mathbb{E} \left[ e^{g(W_1)(\mu - r) - \frac{1}{2}\pi_0^2 \sigma^2 + \pi_0 \sigma W_1} \right] + (1 - \beta) \mathbb{E} \left[ e^{g(W_1)(\mu - r)} \right] \right).$$

Assuming that $\mathbb{E}[g(W_1)] = \pi_0$, it holds that the expected terminal value is increasing in the covariance.\footnote{Recall that the insurance company conducts a constant mix strategy. Based on the assumption that the price dynamics of the risky asset is given in terms of a geometric Brownian motion, the asset side dynamics is also (for a given investment fraction $\pi$) generated by a geometric Brownian motion with volatility $\pi \sigma$.} A further consequence of the dependence structure is that the...
Merton solution approximatively implies the optimal splitting factor only in the case that \( g(W_1) = \pi_0 \) a.s.

We now analyze the optimal splitting factor \( \beta \) which maximizes \( \mathbb{E}[u(V_2)] \) in the presence of different management rules.

**Proposition 3 (Optimal splitting factor)** In the special case that \( \pi_0 = \frac{\mu - r}{\gamma \sigma^2} \), the management rule (ii) (constant investment fraction over whole investment horizon) implies that the optimal splitting factor of the insured is \( \beta = 1 \) (upfront premium). Compared to this, the directional effect of the management rules (i) and (iii) are determined by the sign of

\[
\frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} \bigg|_{\beta=1} = (1 - \gamma) \text{Cov} \left[ u \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right] - e^r \text{Cov} \left[ u' \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right]
\]

Notice that \( u'(x) = x^{-\gamma} > 0 \) and \( u''(x) = -\gamma x^{-\gamma-1} < 0 \). Thus (for \( \tilde{g}' < 0 \)) it holds

\[
\text{Cov} \left[ u \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right] < 0
\]

such that for \( \gamma > 1 \)

\[
(1 - \gamma) \text{Cov} \left[ u \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right] > 0
\]

In addition, it holds (\( u' \) is a decreasing function)

\[
\text{Cov} \left[ u' \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right] > 0
\]

However, the opposite signs are true in the case that \( \mathbb{E}[\tilde{g}(W_1)] = 1 \) and \( \tilde{g}' > 0 \). In both cases, the covariance sign is ambiguous.

We now give further insights based on three cornerstone cases for the management rule function \( g \) as discussed in Remark 1:

- (i) \( g(W_1) = d\pi_0 1_{\{W_1 \leq 0\}} + u\pi_0 1_{\{W_1 > 0\}} \)
- (ii) \( g(W_1) = \pi_0 \)
- (iii) \( g(W_1) = u\pi_0 1_{\{W_1 \leq 0\}} + d\pi_0 1_{\{W_1 > 0\}} \)

Consider e.g. \( \tilde{g}(W_1) = (1 - \epsilon) 1_{\{W_1 > 0\}} + (1 + \epsilon) 1_{\{W_1 \leq 0\}} \). Here, it holds
Certainty equivalents for varying premium fractions

Figure 3: The black line refers to the first case of the management rule function, considered in Remark 1. The gray line refers to \( g(W_1) = \pi_0 \) and the black dashed line refers to the third case of the management rule function. The left (right) figure refers to a level of relative risk aversion \( \gamma = 4 \) (\( \gamma = 2 \)). For the function \( g \) we use the factors \( d = 0.5 \) and \( u = 1.5 \). Notice that the y-axes are not identically scaled.

\[
\frac{\partial \mathbb{E}[u(V_t)]}{\partial \beta} \bigg|_{\beta=1} = \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) e^{(1-\gamma)\left(r + \frac{\tilde{g}(W_1)}{2}\lambda^2\right)} \right]
\]
\[
= e^{(1-\gamma)\left(r + \frac{\tilde{g}(W_1)}{2}\lambda^2\right)} \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) 1_{\{W_1 > 0\}} \right]
\]
\[
+ e^{(1-\gamma)\left(r + \frac{\tilde{g}(W_1)}{2}\lambda^2\right)} \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) 1_{\{W_1 \leq 0\}} \right].
\]

Notice that the investor is 100% in the asset side after \( t = 1 \). In the case that \( \mathbb{E}[\tilde{g}(W_1)] = 1 \) but \( \mathbb{P}(\tilde{g}(W_1) \neq 1) > 0 \), the investor suffers from possible deviations from the optimal Merton solution after \( t = 1 \). Thus, the certainty equivalent of the investor is smaller in the case of a random investment fraction as shown in figure 3.

In order to connect our previous results, we now combine the management rule function of the insurer with a guarantee concept of the insured. Due to option pricing under the pricing measure \( \mathbb{P}^* \), we have to adjust the Brownian motion in the management rule function such that \(^8\)

\[
g(W_1) = g \left( W_1^* - \frac{\mu - r}{\sigma} \right) \quad \text{and} \quad g(W_1^*) = g \left( W_1 + \frac{\mu - r}{\sigma} \right).
\]

\(^8\) Remark that \( W_t^* = \frac{\mu - r}{\sigma} t + W_t \) is a Brownian motion under \( \mathbb{P}^* \).
Throughout the following, recall that $P_{t}^{BS}(S_{t}, T - t, K, \sigma)$ denotes the $t$-price of a European put option with underlying $S$, current asset price $S_{t}$, time to maturity $T - t$ in a Black and Scholes model setup where the dynamics of $S$ are given in terms of a geometric Brownian motion with volatility $\sigma$.

Because of the guarantee costs, we again have to adjust the premium fractions such that $\beta_{0} + \beta_{1} + Put_{0} = 1$ as in the previous section. Depending on $\beta_{0}$ and $\beta_{1}$, the guarantee costs are given by

$$Put_{0} = e^{-2r}E_{P^{*}}\left[\left(G_{2} - \left(\frac{\beta_{0}A_{2}}{A_{0}} + \beta_{1}e^{r}\frac{A_{2}}{A_{1}}\right)\right)^{+}\mid F_{0}\right]$$

We first have a look on the upfront premium case $\beta_{1} = 0$. The guarantee costs at $t = 0$ are given by

$$Put_{0} = e^{-2r}E_{P^{*}}\left[\left(G_{2} - \frac{A_{2}}{A_{0}}\right)^{+}\mid F_{0}\right]$$

$$= \beta_{0}e^{-2r}E_{P^{*}}\left[\left(G_{2} - \frac{A_{1}A_{2}}{A_{0}A_{1}}\right)^{+}\mid F_{0}\right]$$

$$= \beta_{0}e^{-2r}E_{P^{*}}\left[E_{P^{*}}\left[\left(G_{2} - \frac{A_{1}A_{2}}{A_{0}A_{1}}\right)^{+}\mid F_{1}\right]\right]$$

$$= \beta_{0}e^{-r}E_{P^{*}}\left[A_{1}A_{0}e^{-r}E_{P^{*}}\left[\left(G_{2}A_{0} - \frac{A_{2}}{A_{1}}\right)^{+}\mid F_{1}\right]\right].$$

Notice that

$$e^{-r}E_{P^{*}}\left[\left(G_{2}A_{0} - \frac{A_{2}}{A_{1}}\right)^{+}\mid F_{1}\right] = P_{1}^{BS}\left(1,1,K = \frac{G_{2}A_{0}}{\beta_{0}A_{1}}, g(W_{1})\sigma\right),$$

i.e.

$$Put_{0} = \beta_{0}e^{-r}E_{P^{*}}\left[A_{1}A_{0}P_{1}^{BS}\left(1,1,K = \frac{G_{2}A_{0}}{\beta_{0}A_{1}}, g(W_{1})\sigma\right)\right].$$

In particular, the higher $\frac{A_{1}}{A_{0}}$ is, the lower is the strike of the inner option. For $g(w) = \pi_{0}$ ($\tilde{g}(w) = 1$, respectively) the guarantee costs, as seen in the previous section in equation (2), simplify to

$$Put_{0} = \beta_{0}P_{0}^{BS}\left(1,2,K = \frac{G_{2}}{\beta_{0}}, \pi_{0}\sigma\right).$$
Fair β-tuple and certainty equivalents for γ = 2

Figure 4: The black dots refer to the first case of the management rule function, considered in Remark 1. The gray dots refer to \( g(W_1) = \pi_0 \) and the red dots refer to the third case of the management rule function. Both figures refer to \( \pi = \pi^M e^r = 0.5 \) and a level of relative risk aversion \( \gamma = 2 \). For the function \( g \) we use the factors \( d = 0.5 \) and \( u = 1.5 \). The left figure shows combinations of fair \( (\beta_0, \beta_1) \)-tuple. The right figure presents the certainty equivalents depending on the adjusted premium fractions (cf. section 3).

For the postponed premium case \( \beta_0 = 0 \), the guarantee costs at \( t = 0 \) are given by

\[
Put_0 = e^{-2rE^*_{\mathbb{P}}\left[\left(G_2 - \beta_1 e^{r\frac{A_2}{A_1}}\right)^+\bigg|\mathcal{F}_0\right]} = \beta_1 E^*_{\mathbb{P}}\left[e^{-rE^*_{\mathbb{P}}\left[\left(e^{-r\beta_1} \frac{G_2}{A_1} - \frac{A_2}{A_1}\right)^+\bigg|\mathcal{F}_1\right]}\right].
\]

Notice that

\[
e^{-rE^*_{\mathbb{P}}\left[\left(e^{-r\beta_1} \frac{G_2}{A_1} - \frac{A_2}{A_1}\right)^+\bigg|\mathcal{F}_1\right]} = P^{BS}_1 \left(1, 1, K = \frac{G_2}{e^{-r\beta_1}}, g(W_1)\sigma\right)
\]

such that

\[
Put_0 = \beta_1 E^*_{\mathbb{P}}\left[P^{BS}_1 \left(1, 1, K = \frac{G_2}{e^{-r\beta_1}}, g(W_1)\sigma\right)\right].
\]

For \( g(w) = \pi_0 \) (\( \tilde{g}(w) = 1 \), respectively) the guarantee costs simplify to

\[
Put_0 = \beta_1 P^{BS}_1 \left(1, 1, K = \frac{G_2}{e^{-r\beta_1}}, \pi_0\sigma\right).
\]

Because the put is self-financed, we need to determine fair \( (\beta_0, \beta_1) \)-tuple. These tuple are presented in the left-hand side of Figure 4. We also have a closer look at the
expected utility. The right-hand side of Figure 4 shows the certainty equivalents of the insured depending on the adjusted premium fraction \( \frac{\beta_0}{\beta_0 + \beta_1} \). It can be seen that the certainty equivalents of the management rule function \( g(W_1) = \pi_0 \) is higher than the other two cases. This points out that also the choice of the management rule function is highly relevant for calculating the expected utility of the insured.

5 Conclusion

We consider the interactions of the premium contribution scheme, embedded guarantees, and different management rules accounting of a regime switch in the risk profile of the insurance company. Within a Black and Scholes model setup and an investor with constant relative risk aversion, we first analyze the impact of periodic premium contributions on the pricing of participating life insurance contracts (where a guarantee is included) and the utility implied to the insured without a management rule. Thereby, the choice of the optimal premium fractions crucially depends on the level of risk aversion and investment fraction. When taking into account a management rule, we show that this rule also has an impact on the choice of the optimal premium fractions. We shed light on two effects on the utility of the insured. The combination of contribution scheme, embedded guarantees and management rule has an impact on the investment risk of the insured. Assuming that the guarantees are fairly priced, the guarantee costs also depend on all the above mentioned factors. We illustrate our results in pictures and tables. Thus, our paper gives first hints, for both the insurer and insured, for optimal investment behavior.
Appendix

Appendix A: Proof of Lemma 1

\[ e^{-2r}\mathbb{E}^*_{p^*}[\tilde{V}_2] = e^{-2r}\mathbb{E}^*_{p^*}[\beta_0 \frac{A_2}{A_0} + \beta_1 e^r \frac{A_2}{A_1}] \]
\[ = \beta_0 e^{-2r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_0}\right] + \beta_1 e^{-r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_1}\right] \]
\[ = \beta_0 + \beta_1 \]

\[ e^{-2r}\mathbb{E}^*_{p^*}[L_2] = e^{-2r}\mathbb{E}^*_{p^*}[\tilde{V}_2 + (G_2 - \tilde{V}_2)^+] \]
\[ = e^{-2r}\mathbb{E}^*_{p^*}[\tilde{V}_2] + e^{-2r}\mathbb{E}^*_{p^*}[(G_2 - \tilde{V}_2)^+] \]
\[ = \beta_0 e^{-2r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_0}\right] + \beta_1 e^{-r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_1}\right] + Put_0 \]
\[ = \beta_0 + \beta_1 + Put_0 \]

where

\[ e^{-2r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_0}\right] = e^{-2r}\mathbb{E}^*_{p^*}\left[e^{2r-\sigma^2_A e^{\sigma_A \tilde{W}_2}}\right] \]
\[ = e^{-\sigma^2_A}\mathbb{E}^*_{p^*}\left[e^{\sigma_A \tilde{W}_2}\right] \]

and

\[ e^{-r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_1}\right] = e^{-r}\mathbb{E}^*_{p^*}\left[e^{r-\frac{1}{2}\sigma^2_A e^{\sigma_A (\tilde{W}_2 - \tilde{W}_1)}}\right] \]
\[ = e^{-\frac{1}{2}\sigma^2_A}\mathbb{E}^*_{p^*}\left[e^{\sigma_A (\tilde{W}_2 - \tilde{W}_1)}\right] \]

Under the condition that

\[ \sigma_A \tilde{W}_2 \sim N(0, 2\sigma^2_A) \] and \( E[e^x] = e^{\mu + \frac{1}{2}\sigma^2} \), where \( X \sim N(\mu, \sigma^2) \),

it follows

\[ e^{-2r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_0}\right] = e^{-\sigma^2_A}\mathbb{E}^*_{p^*}\left[e^{\sigma_A \tilde{W}_2}\right] \]
\[ = e^{-\sigma^2_A}e^{\frac{1}{2}2\sigma^2_A} \]
\[ = 1 \]

and

\[ e^{-r}\mathbb{E}^*_{p^*}\left[\frac{A_2}{A_1}\right] = e^{-\frac{1}{2}\sigma^2_A}\mathbb{E}^*_{p^*}\left[e^{\sigma_A (\tilde{W}_2 - \tilde{W}_1)}\right] \]
\[ = e^{-\frac{1}{2}\sigma^2_A}e^{\frac{1}{2}\sigma^2_A} \]
\[ = 1. \]
Thus, fair pricing of liabilities \( e^{-2r} \mathbb{E}_p[L_2] = 1 \) is fulfilled if 
\[ \beta_0 + \beta_1 + \text{Put}_0 = 1. \]

**Appendix B: Proof of Proposition 1**

ad a):

\[
\mathbb{E}_p[u(L_2)] = \mathbb{E}_p \left[ u \left( \max \left\{ \frac{A_2}{A_0}, G_2 \right\} \right) \right] 
= \mathbb{E}_p \left[ u \left( \frac{A_2}{A_0} \{ \frac{A_2}{A_0} > G_2 \} + G_2 \{ \frac{A_2}{A_0} \leq G_2 \} \right) \right] 
= \mathbb{E}_p \left[ u \left( \frac{A_2}{A_0} \{ \frac{A_2}{A_0} > G_2 \} \right) \right] + \mathbb{E}_p \left[ u \left( G_2 \{ \frac{A_2}{A_0} \leq G_2 \} \right) \right] 
= E_1 + E_2,
\]

where

\[
E_1 := \mathbb{E}_p \left[ u \left( \frac{A_2}{A_0} \{ \frac{A_2}{A_0} > G_2 \} \right) \right] 
E_2 := \mathbb{E}_p \left[ u \left( G_2 \{ \frac{A_2}{A_0} \leq G_2 \} \right) \right]
\]

Let us first calculate the value of \( E_1 \) and recall, that \( u(x) = \frac{x^{1-\gamma}}{1-\gamma} \).

\[
E_1 = \frac{1}{1-\gamma} \mathbb{E}_p \left[ \left( \frac{A_2}{A_0} \right)^{(1-\gamma)} \{ \frac{A_2}{A_0} > G_2 \} \right] 
= \frac{1}{1-\gamma} \mathbb{E}_p \left[ e^{(2\mu_A-\sigma_A^2)(1-\gamma)+(1-\gamma)\sigma_A W_2} \{ \frac{A_2}{A_0} > G_2 \} \right],
\]

where the last equality holds because \( \frac{A_2}{A_0} = e^{2\mu_A-\sigma_A^2+\sigma_A W_2} \). Now our aim is to use Girsanov’s theorem to calculate the expected value. For this we need the Radon-Nikodym density, which is in our setting given by

\[
Z_2 := e^{-\sigma_A^2(1-\gamma)^2+\sigma_A(1-\gamma)W_2}.
\]

Rewriting (8) we get

\[
E_1 = \frac{1}{1-\gamma} e^{(2\mu_A-\sigma_A^2)(1-\gamma)} e^{(1-\gamma)^2\sigma_A^2} \mathbb{E}_p \left[ e^{-\sigma_A^2(1-\gamma)^2+\sigma_A(1-\gamma)W_2} \{ \frac{A_2}{A_0} > G_2 \} \right] 
= \frac{1}{1-\gamma} e^{(2\mu_A-\sigma_A^2)(1-\gamma)} e^{(1-\gamma)^2\sigma_A^2} \mathbb{E}_p \left[ Z_2 \{ \frac{A_2}{A_0} > G_2 \} \right].
\]
With Girsanov’s Theorem,

\[
\mathbb{E}_\mathbb{P} \left[ Z_2 \mathbb{1}_{\left\{ \frac{A_2}{A_0} > G_2 \right\}} \right] = \tilde{\mathbb{P}} \left( \frac{A_2}{A_0} > G_2 \right),
\]

where \( \tilde{\mathbb{P}} \) is the uniquely determined equivalent martingale measure of \( \mathbb{P} \) and \( \tilde{W}_T \) is a BM under \( \tilde{\mathbb{P}} \) given by \( \tilde{W}_2 = W_2 - 2(1 - \gamma)\sigma_A \). Using this result, we get

\[
\tilde{\mathbb{P}} \left( \frac{A_2}{A_0} > G_2 \right) = 1 - \tilde{\mathbb{P}} \left( \frac{A_2}{A_0} \leq G_2 \right) = 1 - \tilde{\mathbb{P}} \left( \frac{\tilde{W}_2}{\sqrt{2}} \leq \frac{\ln(G_2) - 2[\mu_A - (\gamma - \frac{1}{2})\sigma_A^2]}{\sqrt{2} \sigma_A} \right)
\]

\[
= 1 - \Phi \left( \frac{\ln(G_2) - 2[\mu_A - \sigma_A^2(\gamma - \frac{1}{2})]}{\sqrt{2} \sigma_A} \right). \quad (10)
\]

Combining (9) and (10) it holds

\[
E_1 = \frac{1}{1 - \gamma} e^{(1-\gamma)(2\mu_A - \gamma\sigma_A^2)} \left\{ 1 - \Phi \left( \frac{\ln(G_2) - 2[\mu_A - \sigma_A^2(\gamma - \frac{1}{2})]}{\sqrt{2} \sigma_A} \right) \right\}. \quad (11)
\]

It remains to calculate \( E_2 \):

\[
E_2 = \frac{1}{1 - \gamma} G_2^{(1-\gamma)} \mathbb{E}_\mathbb{P} \left[ 1 \left\{ \frac{A_2}{A_0} \leq G_2 \right\} \right] = \frac{1}{1 - \gamma} G_2^{(1-\gamma)} \mathbb{P} \left( \frac{A_2}{A_0} \leq G_2 \right)
\]

\[
= \frac{1}{1 - \gamma} G_2^{(1-\gamma)} \Phi \left( \frac{\ln(G_2) - 2[\mu_A + \sigma_A^2]}{\sqrt{2} \sigma_A} \right) \quad (12)
\]

Combining (11) and (12) we get the final solution.

For the CE just use the relation \( CE = u^{-1}(\mathbb{E}_\mathbb{P}[u(L_2)]) \) with \( u^{-1}(x) = ((1 - \gamma)x)^{\frac{1}{1-\gamma}} \).

ad (b):

\[
\mathbb{E}_\mathbb{P}[u(L_2)] = \mathbb{E}_\mathbb{P} \left[ u \left( \max \left\{ \frac{e^{rA_2}}{A_1}, G_2 \right\} \right) \right]
\]

\[
= \mathbb{E}_\mathbb{P}[E_3] + \mathbb{E}_\mathbb{P}[E_4],
\]

where

\[
E_3 := \frac{1}{1 - \gamma} \mathbb{E}_\mathbb{P} \left[ \left( \frac{e^{rA_2}}{A_1} \right)^{1-\gamma} 1_{\left\{ \frac{e^{rA_2}}{A_1} > G_2 \right\}} \right]
\]

\[
E_4 := \frac{1}{1 - \gamma} \mathbb{E}_\mathbb{P} \left[ (G_2)^{1-\gamma} 1_{\left\{ \frac{e^{rA_2}}{A_1} \leq G_2 \right\}} \right].
\]
Lets calculate $E_3$, where we use that $\frac{A_2}{A_1} = e^{\mu A - \frac{1}{2} \sigma_A^2 + \sigma_A (W_2 - W_1)}$. So it holds

$$E_3 = \frac{1}{1 - \gamma} \mathbb{E}_\hat{P} \left[ (e^{r A_2 / A_1})^{1 - \gamma} 1 \{e^{A_2 / A_1} > G_2\} \right]$$

$$= \frac{1}{1 - \gamma} e^{r(1 - \gamma) \left(\mu_A - \frac{1}{2} \gamma \sigma_A^2\right)} \mathbb{E}_\hat{P} \left[ e^{\sigma_A (1 - \gamma) (W_2 - W_1)} 1 \{e^{A_2 / A_1} > G_2\} \right]$$

(13)

Like in the proof of part (a) we now want to use Girsanov’s Theorem. The Radon-Nikodym density here is given by

$$\hat{Z}_2 := e^{-\frac{1}{2} (1 - \gamma)^2 \sigma_A^2 + \sigma_A (1 - \gamma) (W_2 - W_1)}.$$

Rewriting (13) and with the fact that

$$\sigma_A (\hat{W}_2 - \hat{W}_1) = \sigma_A (W_2 - W_1) - \sigma_A^2 (1 - \gamma),$$

where $\hat{W}_T$ is a BM under the uniquely determined equivalent martingale measure $\hat{P}$, we get

$$E_3 = \frac{1}{1 - \gamma} e^{r(1 - \gamma) \left(\mu_A - \frac{1}{2} \gamma \sigma_A^2\right)} \mathbb{E}_\hat{P} \left[ \hat{Z}_2 1 \{e^{A_2 / A_1} > G_2\} \right]$$

$$= \frac{1}{1 - \gamma} e^{r(1 - \gamma) \left(\mu_A - \frac{1}{2} \gamma \sigma_A^2\right)} \left\{ 1 - \hat{P} \left( e^{A_2 / A_1} \leq G_2 \right) \right\}$$

$$: = \frac{1}{1 - \gamma} e^{(1 - \gamma) (r + \mu_A - \frac{1}{2} \gamma \sigma_A^2)} \left\{ 1 - \Phi \left( \frac{\ln(G_2) - r - \mu_A - \frac{1}{2} \gamma \sigma_A^2}{\sigma_A} \right) \right\}$$

(14)

It remains to calculate $E_4$:

$$E_4 = \frac{1}{1 - \gamma} \mathbb{E}_\hat{P} \left[ (G_2)^{1 - \gamma} 1 \{e^{A_2 / A_1} \leq G_2\} \right]$$

$$= \frac{1}{1 - \gamma} G_2^{(1 - \gamma)} \hat{P} \left( e^{A_2 / A_1} \leq G_2 \right)$$

$$= \frac{1}{1 - \gamma} G_2^{(1 - \gamma)} \left( W_2 - W_1 \leq \frac{\ln(G_2) - r - \mu_A + \frac{1}{2} \sigma_A^2}{\sigma_A} \right)$$

$$= \frac{1}{1 - \gamma} G_2^{(1 - \gamma)} \Phi \left( \frac{\ln(G_2) - r - \mu_A + \frac{1}{2} \sigma_A^2}{\sigma_A} \right).$$

(15)

Combining (14) and (15) we get the final result.

The certainty equivalent can be calculated analogously to part (a). □
Proof of Proposition 3

Thereby, we continue with the calculations of equation (6) in the previous chapter. First, we consider case (ii). In the special case that \( \pi_0 = \frac{\mu - r}{\gamma \sigma^2} \), the investor can obtain the Merton solution by choosing \( \beta = 1 \), i.e. by choosing an upfront contribution of 100% of the present premium value. Notice that

\[
\mathbb{E}[u(V_2)] = \mathbb{E} \left[ u \left( \beta \frac{A_2}{A_0} + (1 - \beta) e^r \frac{A_2}{A_1} \right) \right] \\
= \mathbb{E} \left[ \mathbb{E} \left[ u \left( \beta \frac{A_1}{A_0} + (1 - \beta) e^r \frac{A_2}{A_1} \right) \bigg| \mathcal{F}_1 \right] \right].
\]

Using \( u(x) = \frac{x^{1-\gamma}}{1-\gamma} \) gives

\[
\mathbb{E}[u(V_2)] = \mathbb{E} \left[ \mathbb{E} \left[ \left( \frac{A_2}{A_1} \right)^{1-\gamma} u \left( \beta \frac{A_1}{A_0} + (1 - \beta) e^r \right) \bigg| \mathcal{F}_1 \right] \right] \\
= \mathbb{E} \left[ u \left( \beta \frac{A_1}{A_0} + (1 - \beta) e^r \right) \mathbb{E} \left[ \left( \frac{A_2}{A_1} \right)^{1-\gamma} \bigg| \mathcal{F}_1 \right] \right] \\
= \mathbb{E} \left[ u \left( \beta \frac{A_1}{A_0} + (1 - \beta) e^r \right) e^{(1-\gamma)(r + g(W_1)(\mu - r) - \frac{1}{2} \gamma g^2(W_1)\sigma^2)} \right]
\]

Let \( \lambda := \frac{\mu - r}{\sigma} \). For \( \pi_0 = \frac{\mu - r}{\gamma \sigma^2} = \frac{\lambda}{\gamma} \) it holds

\[
\frac{A_1}{A_0} = e^{r + \frac{\lambda^2}{\gamma} + \frac{1}{2} W_1 - \frac{1}{2} \frac{\lambda^2}{\gamma}}
\]

Let \( \tilde{g}(W_1) := \frac{\sigma}{\lambda} g(W_1) \), then

\[
r + g(W_1)(\mu - r) - \frac{1}{2} \gamma g^2(W_1)\sigma^2 = r + \tilde{g}(W_1) \lambda^2.
\]

Now notice that

\[
\frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} |_{\beta=1} = \mathbb{E} \left[ u'( \beta \frac{A_1}{A_0} + (1 - \beta) e^r ) \left( \frac{A_1}{A_0} - e^r \right) e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right].
\]

Using \( u'(x) = x^{-\gamma} \) gives

\[
\frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} |_{\beta=1} = \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \right].
\]

First, consider \( \tilde{g}(W_1) = 1 \). Here, it holds

\[
\frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} |_{\beta=1} = e^{(1-\gamma)(r + \frac{\tilde{g}(W_1)}{2\gamma} \lambda^2)} \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) \right]
\]

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and
\[ \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) \right] = \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{1-\gamma} - e^r \left( \frac{A_1}{A_0} \right)^{-\gamma} \right]. \]

Using
\[ \frac{A_1}{A_0} = e^r + \frac{\lambda^2}{\gamma} + \frac{\lambda^2}{\gamma} W_1 - \frac{1}{2} \left( \frac{\lambda}{\gamma} \right)^2 \]

immediately gives \( \mathbb{E}[e^{aW_1}] = e^{\frac{a^2}{2}} \):
\[ \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{1-\gamma} \right] = e^{(1-\gamma)(r + \frac{1}{2} \frac{\lambda^2}{\gamma})} \]
and \( e^r \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \right] = e^{(1-\gamma)(r + \frac{1}{2} \frac{\lambda^2}{\gamma})}, \)
i.e. for \( \tilde{g}(W_1) = 1 \) it holds
\[ \frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} \bigg|_{\beta=1} = 0, \]
such that the first order condition holds for \( \beta = 1. \)

Now, consider the management rule function \( \tilde{g} \) where \( \mathbb{E}[\tilde{g}(W_1)] = 1 \) and \( \tilde{g}' < 0. \)
Notice that with \( \mathbb{E}[XY] = \mathbb{E}[X] \mathbb{E}[Y] + \text{Cov}[X,Y] \) (and \( \mathbb{E}[X] = 0 \)) it follows
\[ \frac{\partial \mathbb{E}[u(V_2)]}{\partial \beta} \bigg|_{\beta=1} = \mathbb{E} \left[ \left( \frac{A_1}{A_0} \right)^{-\gamma} \left( \frac{A_1}{A_0} - e^r \right) e^{(1-\gamma)(r + \frac{\delta(W_1)}{2\gamma \lambda^2})} \right] \]
\[ = \text{Cov} \left[ \left( \frac{A_1}{A_0} \right)^{1-\gamma} - e^r \left( \frac{A_1}{A_0} \right)^{-\gamma}, e^{(1-\gamma)(r + \frac{\delta(W_1)}{2\gamma \lambda^2})} \right] \]
\[ = \text{Cov} \left[ (1 - \gamma) u \left( \frac{A_1}{A_0} \right) - e^r u' \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\delta(W_1)}{2\gamma \lambda^2})} \right] \]
\[ = (1 - \gamma) \text{Cov} \left[ u \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\delta(W_1)}{2\gamma \lambda^2})} \right] - e^r \text{Cov} \left[ u' \left( \frac{A_1}{A_0} \right), e^{(1-\gamma)(r + \frac{\delta(W_1)}{2\gamma \lambda^2})} \right] \]

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References


