Optimal Collateralization with Bilateral Default Risk

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Abstract

We consider over-the-counter (OTC) transactions with bilateral default risk, and study the optimal design of the Credit Support Annex (CSA). In a setting where agents have access to a trading technology, default penalties and collateral costs arise endogenously as a result of foregone investment opportunities. We show how the optimal CSA trades off the costs and volatility of the collateralization procedure against the reduction in exposure to counterparty risk and expected default losses. The results are used to provide insights on the key drivers of different types of collateral rules, including hedging motives, re-hypothecation of collateral, and close-out conventions.

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1 Introduction

The relevance of counterparty risk\textsuperscript{1} for the structuring and pricing of financial transactions has become apparent during the recent financial crisis. The master agreement of a derivative transaction, which a few years ago would have been mainly of interest to the legal department rather than to a structuring desk, is now an integral part of the design and pricing of a deal. At a higher level, the mark-to-market gains or losses arising from counterparty risk\textsuperscript{2} have a significant impact on the earning results of financial institutions, and are therefore systematically monitored and hedged. Similarly, a variety of discount curves is now used for the valuation of assets and liabilities, to reflect the different liquidity and counterparty risk profiles of different overnight indices and tenors.\textsuperscript{3} The new Dodd-Frank regulation in the US, and EMIR provisions in Europe, all emphasize the role of credit risk mitigation tools such as collateralization and segregation of collateral to limit the exposure of OTC markets to systemic risk. Despite the importance of these issues, little theoretical guidance is available to understand the design of the Credit Support Annex (CSA), to justify the form of specific collateral rules, and to quantify the costs and benefits associated with collateralization.

In this article, we consider two credit-risky counterparties that are homogeneous with respect to risk preferences, credit quality, and investment opportunities, but have opposite exposure to a random payoff that is nontradable and uspanned by the tradeable assets. This is a prototypical situation where a risk sharing agreement allows agents to improve their position. There is catch, however: the credit riskiness of the counterparties

\textsuperscript{1}Basel II (2006, Annex 4) defines counterparty risk as ‘the risk that the counterparty to a transaction could default before the final settlement of the transaction’s cash flows’.

\textsuperscript{2}See Brigo et al. (2012b) for an extensive treatment of Credit/Debit Valuation Adjustment (CVA/DVA) and extensions.

\textsuperscript{3}The interbank market now quotes very diverse credit and liquidity premia, even for plain vanilla products.
means that any risk sharing arrangement will be exposed to counterparty risk. On the other hand, any counterparty risk mitigation tool (such as collateralization) will divert resources away from profitable trading opportunities, and hence give rise to an opportunity cost of posting collateral. Even if agents have symmetric collateral requirements, the costs of collateralization need to be weighted against the expected reduction in default penalties that collateralization is able to deliver. Moreover, the collateralization procedure will result in the illiquid exposure being regularly marked-to-market, therefore exposing the agents’ wealth to an additional source of risk, relative to the case of no risk sharing, or risk sharing without collateralization. We develop a model that allows us to formalize these issues and quantify the main trade-offs at play, therefore providing tools that can be used to answer a number of interesting questions. First, we provide a rational for collateral rules commonly observed in OTC markets by endogenizing the collateral rules as a result of the agents’ hedging demand against expected default penalties. Second, by endogenizing both the CSA and the cost of posting collateral, we are able to demonstrate why funding costs and CSA should be taken into account in the valuation of OTC transactions. Third, we provide a robust framework to support the analysis of relevant policy issues, such as the potential impact of mandating suboptimal collateralization levels and moving to centralized clearing for OTC derivates.

Our contribution is related to at least three different strands of literature. We contribute to the analysis of bilateral default risk, initiated in Rendleman (1992) and Duffie and Huang (1996), by explicitly allowing for collateral when defining default losses and valuing OTC instruments. Several contributions have recently looked at bilateral default risk (e.g., Assefa et al., 2010; Brigo et al., 2011; Crépey, 2012a,b, among others) and CSA pricing (e.g., Biffis et al., 2011; Brigo et al., 2012a,b), but they all take collateral rules as exogenously given. Here, we explicitly look at the design of the CSA, and
obtain a number of empirical predictions that shed light on counterparty risk mitigation
tools used in practice.

We also contribute to the discussion on whether funding costs should be part of the
valuation of collateralized products. In the interest-rate swaps market, for example, it
has long been known that traditional swaps valuation formulae based on par bond rates
of a LIBOR-quality issuer (e.g., Duffie and Singleton, 1997) present a fundamental in-
consistency, as interest rate swaps are typically fully collateralized (e.g., ISDA, 2010a)
and hence counterparty risk is negligible.\footnote{This is why econometric models of interest rate swap spreads may assume interest rate swaps to be free of credit risk (e.g., Collin-Dufresne and Solnik, 2001).} Johannes and Sundaresan (2007) find evidence of costly collateral in interest rate swaps by comparing swap market data with
swap values based on portfolios of futures and forward contracts, and by estimating a
dynamic term structure model by using Treasury and swap data. Brigo \emph{et al.} (2012b)
develop a framework to price OTC instruments in the presence of both collateraliza-
tion and funding costs. As opposed to these studies, in our model collateral costs arise
endogenously as the opportunity cost of diverting resources from profitable trading op-
portunities. Even if collateral requirements are symmetric for the counterparties, we
show how the cost of posting collateral does enter the optimal design of the CSA and
the valuation of collateralized transactions.

Finally, we offer relevant tools to quantify the potential impact of centralized clear-
ing on the market for OTC derivatives (Singh and Aitken (2009); Heller and Vause
(2012); Sidanius and Zikes (2012)). Our framework allows one to quantify the utility
gains/losses to the counterparties when using different collateral rules, close-out conven-
tions and collateral treatment rules. Rather than abstracting from variation margins
and reducing the costs of central clearing with the initial margin required by a clearing
house, we demonstrate how variation margins affect the agents’ positions and indirect utilities, thereby generating a possible cost even if the counterparties face symmetrical collateral requirements.

The paper is structured as follows. In the next section, we outline a simple continuous-time model with two agents and two sources of risk, one tradeable and one illiquid. We then provide some results for the private valuation of the illiquid position by agents with CARA preferences. In section 3, we introduce a risk sharing arrangement that mitigates the risk associated with the illiquid position, but gives rise to counterparty risk. We consider the case of no collateral, and fixed or contingent collateralization rules, offering insights into how the CSA can be optimally designed. Section 4 discusses some technical, yet fundamental, issues related to collateral rehypothecation, close-out conventions, and mark-to-market/model approaches to computing collateral amounts. Finally, section 5 concludes, while an appendix collect proofs and additional technical results.

2 Model

Consider two risk-averse agents, denoted by $A$ and $B$, who have opposite exposure to a source of risk $Z$ at time $T > 0$. Assume, for example, that party $A$ is exposed to the payment of $k > 0$ units of the random amount $Z_T$ (a liability equal to $-kZ_T$), whereas party $B$ is in the opposite situation, and expects a random inflow of amount $kZ_T$. For simplicity, let $Z$ have dynamics

$$dZ_t = \sigma_Z dB_t^Z, \quad Z_0 = 0,$$

where $\sigma_Z$ is the volatility of $Z$.
with $B^Z$ a standard Brownian motion and $\sigma_Z$ a positive volatility coefficient. Agents have access to a financial market, where available for trade are a money market account yielding the riskless rate $r > 0$, and a risky security with price $S$ evolving according to

$$dS_t = \mu S_t dt + \sigma_S S_t dB^S_t, \quad S_0 = 1,$$

with $\mu > r$ and $\sigma_S > 0$. We assume that the Brownian motions $B^S, B^Z$ are uncorrelated, meaning that i) tradable assets offer agents no spanning opportunities for $Z$, and ii) the exposures $\pm k Z_T$ are fully illiquid, as they give rise to no intermediate dividends and the price of traded securities offer no information on their market value. The presence of a nontradeable source of risk makes the market incomplete, and we need to take a stance on agents’ preferences to identify a valuation functional consistent with the absence of arbitrage opportunities. We assume agents to have both CARA utility $u(x) = -\frac{1}{\gamma} e^{-\gamma x}$, with $\gamma > 0$, and to optimize their utility from terminal wealth. Agents are credit risky: we model their default exogenously, as triggered by the first jumps of two Poisson processes with the same parameter $\lambda > 0$. The use of a reduced form approach to default is consistent with the fact that we will focus on marginal transactions having no direct impact on the agents’ default probability. We have abstracted from agents’ heterogeneity (safe for the opposite exposure to $Z_T$) to keep the paper focused and emphasize the role of counterparty risk and its mitigation. Denoting by $\tau^i$ the default time of party $i \in \{A, B\}$, and by $N_i = 1_{r^i \leq t}$ the default indicator process, we can express the dynamics of each agent’s wealth, $W^i_t$, as

$$dW^i_t = (1 - N^i_{t-}) \left[ (r W^i_t + \pi^i_t (\mu - r)) dt + \pi^i_t \sigma_S dB^S_t \right] + N^i_t W^i_t r dt, \quad W^i_0 = w^i. \quad (2.1)$$
In the above, \( \pi_i^t \) denotes the amount allocated by each agent to the risky asset at each time \( t \in [0, T] \), and it is assumed that conditional on default the residual wealth is invested in the money market account until maturity (see e.g. Alvarez and Jermann (2000)). Consistently with our assumption of exogenous default, we interpret the agents’ wealth as the resources dedicated to a specific transaction, and hence as the balance of a trading account, which can turn negative and hence attract interest rate charges. For simplicity, we assume the latter to accrue at rate \( r \).

Because \( Z \) has symmetric distribution, both agents will solve the problem

\[
\max_{\pi^i} E \left[ u(W_i^T - kZ_T) \right], \quad (2.2)
\]

subject to the budget constraint (2.1). The optimal investment strategies and value functions are then given in the following proposition.

**Proposition 2.1.** On \( \{ \tau^i > t \} \), the agents’ optimal value functions are given by

\[
v(t, W_i^t, Z_t) = -\frac{1}{\gamma} \exp \left( -\gamma W_i^t e^{r(T-t)} + \gamma k Z_t + \frac{1}{2} \gamma^2 k^2 \sigma_S^2 (T - t) \right) \\
\left( (1 - \alpha) \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2} (T - t) - \lambda (T - t) \right) + \alpha \right), \quad (2.3)
\]

with \( \alpha = \frac{\lambda}{\lambda + \frac{1}{2} \frac{(\mu - r)^2}{\sigma_S^2}} \in (0, 1) \) and \( i \in \{ A, b \} \). The optimal allocation to the risky asset is given by

\[
\pi_i^{*,*} = \exp \left( -r(T - t) \right) \frac{\mu - r}{\gamma \sigma_S^2}. \quad (2.4)
\]

The above shows that the optimal allocation to the risky asset is unaffected by the exposure to the illiquid payoff \( kZ \) and to default risk, as they are both unspanned by
the tradeable assets. As $\lambda$ goes to zero, $\alpha$ vanishes, and expression (2.3) reduces to

$$v(t, W^i_t, Z_t) = \frac{-1}{\gamma} \exp \left( -\gamma W^i_t e^{r(T-t)} + \gamma kZ_t + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) \right) \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} (T-t) \right),$$

showing how the agents’ indirect utility is affected by the illiquid exposure. As demonstrated by Svensson and Werner (1993) and Teplà (2000), in the above we can identify the way each agent marks to market her illiquid position. Denoting by $V^i_t$ agent $i$’s private valuation of the nontradeable exposure $(1_{i=B} - 1_{i=A})kZ_T$, we can rewrite (2.5) as

$$v(t, W^i_t, Z_t) = \frac{-1}{\gamma} \exp \left( -\gamma e^{r(T-t)} (W^i_t + (1_{i=B} - 1_{i=A})V^i_t) \right) \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} (T-t) \right),$$

where the private valuation takes the explicit form

$$V^i_t = E^{i,*}_t \left[ e^{-r(T-t)}(1_{i=B} - 1_{i=A})kZ_T \right] = e^{-r(T-t)} \left( (1_{i=B} - 1_{i=A})kZ_t + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) \right).$$

(2.7)

Here, the conditional expectation $E^{i,*}_t[\cdot]$ represents the no-arbitrage pricing functional supported by agent $i$’s preferences, given the available trading opportunities and budget constraint (2.1). As the agents have opposite exposure to $kZ_T$, from (2.5)-(2.7) we see that the functional form of the agents’ indirect utility will be the same. The agents’ private valuation will allow us to properly address the definition of collateralization and close-out conventions in OTC products subject to bilateral default risk.

Having clarified the structure of the solution for the no default risk case, we can now address the question of the impact of default risk, i.e., $\lambda > 0$. From the second line in
(2.3), we see that the value function discounts the cost of being excluded from the capital market following the default event. The default penalty is endogenous and shaped by i) the remaining time horizon over which agents are exposed to both illiquidity and default risk, and ii) the forgone investment opportunities, as summarized by the Sharpe ratio $\frac{\mu - r}{\sigma}$. This will be an important component of the results appearing in the following sections.

The baseline case just outlined seems a bit restrictive. Because of risk aversion and the opposite exposure to $Z_T$, there are natural risk sharing opportunities for agents A and B. As a simple example, consider the case of a forward contract, i.e., an agreement to exchange the amount $Z_T$ at time $T$ for a fixed price $p$ agreed today. By symmetry of the agents' positions and the distribution of $Z$, we set $p = 0$ in the following. The spanning opportunities offered by the forward contract come at a cost: as agents can default on their obligations, the risk sharing arrangement gives rise to counterparty risk. Moreover, the contract will only mitigate the exposure to $Z$ at the terminal date $T$, but will not solve the illiquidity problem over $(0, T)$. We will see that designing a CSA meeting the needs of both parties will have the effect of mitigating both counterparty risk and illiquidity, as collateral flows will play the role of intermediate dividends paid by the illiquid position in $Z$. Before studying this situation in detail in the following section, we note that in the presence of the forward contract and no default risk, the optimization problem (2.2) and expressions (2.5)-(2.6) would still apply with $k$ replaced by $k - 1$. 

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3 Counterparty risk

As each counterparty can default on its obligations, the agents’ budget constraints will take into account both their own risk of default and the risk of the counterparty not being able to honor the risk sharing agreement. In the presence of a CSA indicating how to exchange collateral amounts, the budget constraints will also take into account the cashflows associated with collateral or other counterparty risk mitigation tools (put clauses, cancellability options, etc.). Let \( \tau = \min(\tau^A, \tau^B) \) denote the default time of the forward contract, and \( N_t := 1_{\tau \leq t} \) the associated indicator process. We denote by \( C^i \) the collateral process as seen from party \( i \), with the understanding that \( C^A = -C^B \).

We assume collateral to be posted continuously and in cash. We let \( C_0 \) be zero, and require \( C_t \) to be measurable with respect to the information available to agents an instant before each time \( t > 0 \), in particular before a default may occur. Collateral amounts enter and exit the agents’ trading account, depending on the relative exposure of each agent to the counterparty and on the definition of collateral indicated by the CSA. Collateral amounts are fully fungible, in the sense that they can be optimally invested by the agent in the financial market. This means that there is an opportunity cost associated with posting collateral: this cost is endogenous in our model and related to the investment opportunities available in the market. The budget constraint (2.1) now now the following form:\(^5\)

\[
dW^i_t = (1 - N^i_t) \left[ (rW^i_t + \pi^i_t(\mu - r)) \, dt + \pi^i_t \sigma S_t \, dB^S_t \right] \\
+ N^i_t W^i_t r \, dt + (1 - N^i_t) \left[ dC^i_t + \left( (R^t_{i-})^+ - C^+_t \right) \, dN^i_t - \left( (R^t_{i-})^- - C^-_t \right) \, dN^j_t \right],
\]

\hspace{1cm}(3.1)

\(^5\)We denote \( a^+ = \max\{0, a\} \) and \( a^- = \max\{-a, 0\} \) for \( a \in \mathbb{R} \).
with $W_0^i = w^i$, $i, j \in \{A, B\}$, $i \neq j$. In the above, $R^i$ denotes the replacement cost of the forward contract for agent $i$. The idea is that conditional on party $i$ ($j$) defaulting, agent $i$ would receive (pay) the full replacement cost of the forward contract from (to) the surviving counterparty, net of any collateral already held (posted). Note that in case of overcollateralization, i.e., $C^\pm > R^\pm$, provision of the full replacement cost may entail receiving any excess collateral posted. For ease of exposition, in this section we consider a common risk-free (and risk-neutral) close-out convention (e.g., Brigo et al., 2012b), meaning that we set

$$R^i_t = E_t \left[ e^{-r(T-t)}(\pm Z_T) \right] = \pm e^{-r(T-t)} Z_t.$$ \hspace{1cm} (3.2)

Alternative close-out conventions, as well as the possibility to endogenize the replacement cost, are discussed in sections 4. The general case is treated in the appendix. By symmetry of $Z$, we have that both agents solve now the problem

$$\max_{(x^i, C^i)} E \left[ u(W^i_T + (1-k)Z_T) \right],$$ \hspace{1cm} (3.3)

subject to the budget constraint (3.1). We now solve the problem at different levels of generality. We first study the case of no collateral, to understand the impact of counterparty risk (in addition to default risk and illiquidity) on the agents’ positions. We then consider two classes of admissible CSAs: those specifying fixed fixed collateral rules, and those indicating contingent collateral rules.
3.1 No collateral

As there is counterparty risk, in addition to each agent’s own default risk, the valuation of an OTC product at time $t$ from the point of view of agent $i \in \{A, B\}$ gives rise to three relevant cases:

- Agent $i$ has already defaulted, $\{\tau^i \leq t < \tau^j\}$.
- The counterparty has already defaulted, $\{\tau \leq t < \tau^i\}$.
- No default yet, $\{\tau > t\}$.

The first case is uninteresting, as agent $i$’s wealth is simply invested in the money market account and any resources available at the terminal date $T$ will be used to meet the random exposure to $Z_T$. In the second case, the agent is faced with the exposure $kZ_T$, without the help of a hedging instrument, and we are back to the case covered in Proposition 2.1. The third case is more interesting, as the indirect utility of each agent will reflect the trade-off between the risk sharing benefits of the forward contract against the counterparty risk it gives rise to.

**Proposition 3.1.** Consider problem (3.3), with the collateral strategy restricted to the null process. On the event $\{\tau > t\}$, the optimal value function of agent $i$ is given by

$$v(t, W^i_t, Z_t) = -\frac{1}{\gamma} \exp \left( -\gamma W^i_t e^{(T-t)} + \gamma kZ_t + \frac{1}{2} \gamma^2 k^2 \sigma_Z^2 (T-t) \right) \exp (-2\lambda t) \lambda \Theta(t, Z_t),$$

where $\Theta$ solves PDE (A.1) reported in the appendix and admitting an explicit solution. The optimal allocation to the risky asset is again given by (2.4).
Comparing expression (3.4) with (2.3), we see that the agents’ indirect utility discounts the expected counterparty risk losses (as opposed to the individual default penalty only), by taking into account the overall cost of the first default event, should it occur over the next time interval. With probability \( \exp(-2\lambda t) \lambda \), the first default will occur over \((t, t + dt)\) and generate the utility gains/losses captured by \( \Theta(t, Z_T) \). Utility losses will include the (expected) default penalties discussed in the previous section and any outflows needed to provide the replacement cost. Utility gains will include any receivables due in case the agent is in-the-money.

3.2 Fixed collateralization

Consider the fully-fledged optimization problem (3.3), where collateral is forced to be expressed as a deterministic fraction of the risk-free close-out price (3.2). The space of admissible collateral rules is restricted to measurable functions \( c^i : t \in [0, T] \rightarrow \mathbb{R}_+ \) satisfying \( c^i(0) = 0 \) and \( c^i = -c^j, i, j \in \{A, B\} \). Hence the collateral amount held/posted by agent \( i \) at time \( t \) can be expressed as

\[
C^i_t = c(t)e^{-r(T-t)}Z_t.
\]

It turns out that in this case problem (3.3) can be reduced to a deterministic control problem:

**Proposition 3.2.** Problem (3.3) reduces to

\[
\max_{c(t)} V_0,
\]
where

\[ V_0 = \frac{-1}{\gamma} \exp \left( -\gamma w_0 e^{-\gamma T} + \frac{1}{2} \gamma^2 k^2 \sigma^2_T \right) \]

\[ \int_0^T e^{-\lambda s} \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_T} s + \gamma^2 r^2 \sigma^2_T \int_0^s e^{-r(T-u)} c_u \int_u^T e^{-r(T-w)} c_w \, dw \, du \right) \]

\[ \times \exp \left( -\gamma^2 \sigma^2_T k \int_0^T e^{-r(T-u)} c_u \, du \right) \]

\[ \left( \exp \left( \gamma^2 \sigma^2_T \frac{1}{2} e^{2r(T-s)} s - k \sigma r \int_0^s e^{-r(T-u)} c_u \, du \right) + r \int_0^s e^{-r(T-u)} c_u \, du e^{r(T-s)} \right) \]

\[ \left[ h(s) + (1 - h(s)) \Phi \left( \gamma \sigma Z k \sqrt{\frac{2 \sigma r}{\sigma^2}} \int_0^s e^{-r(T-u)} c_u \, du - \gamma \sigma Z e^{r(T-s)} \sqrt{\frac{2 \sigma r}{\sigma^2}} \right) \right] \]

\[ + e^{-2\lambda T} \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_T} T + \frac{1}{2} \gamma^2 \sigma^2_T \frac{1}{2} (1 - 2k) \right) \]

\[ \times \exp \left( \gamma^2 r^2 \sigma^2_T \int_0^T e^{-r(T-u)} c_u \int_0^u e^{-r(T-w)} c_w \, dw \, du + (1 - k) \gamma^2 \sigma^2_T r \int_0^s e^{-r(T-u)} c_u \, du \right) \]

\[ \Phi(\cdot) is the distribution function of the central Normal distribution, and \]

\[ h(s) = \alpha + (1 - \alpha) \exp \left( -\frac{1}{2} \frac{(\mu - r)^2}{\sigma^2_T} + \lambda \right) (T - s) \].

To obtain an intuition about this optimization problem, it is useful to consider some special cases.

**Corollary 3.3.** For \( r = 0 \) and \( h(s) \to 1 \), the optimal collateral fraction is: \( c^*(t) = k \).

Hence, we obtain a positive collateral level: due to the agents’ hedging demand, collateralization serves as a mitigation device for counterparty risk. In fact, if the risk sharing agreement fully hedges them against their non-traded exposure, that is if \( k = 1 \), we obtain full collateralization.

**Corollary 3.4.**

1. For \( h(s) \to 1 \) and \( r > 0 \), the optimal collateral fraction \( c^*(t) < k \).

2. For \( r = 0 \), the optimal collateral fraction \( c^*(t) < k \).
Corollary 3.4 indicates that there exist two aspects that push the optimal collateral level away from full collateralization. On the one hand, if interest rates are positive, intermediate collateral flows yield a risky terminal wealth level—and this risk cannot be mitigated in the present case as we assume that the collateral fraction is deterministic. Consider for instance a scenario where neither of the parties defaults and \( k = 1 \): Then there is no risk originating from the transaction, but positive collateral flows still make the parties susceptible to risk.

On the other hand, even if the interest rate is zero, agents evaluate scenarios where collateral is paid differently than scenarios where collateral is received. More precisely, the party receiving the collateral still has the opportunity to invest in the financial market, whereas the defaulting party no longer has access to profitable investment opportunities. Thus marginal utilities differ in scenarios where collateral is paid relative to scenarios where collateral is received, rendering full collateralization suboptimal—even in the symmetric setting considered here.

As an example, we provide some illustrations for the case of a fixed collateral rule \( c^i(t) = c \geq 0 \). From figure 1, we see that the expected utility from terminal wealth is maximized for a strictly positive \( c^* = 48\% \). Figure 2 shows the sensitivity of the optimal collateral fraction with respect to the risk aversion parameter \( \gamma \): the higher \( \gamma \), the higher the collateral fraction (which is always bounded away from 100\%). To gain further insights on the possibility of full collateralization, we let \( k \) vary between zero and one: figure 3 shows that for a low hedging demand, there is no collateralization, whereas for a high hedging demand collateralization is positive but less than full.
Figure 1: Utility level as a function of $c$. Parameter values: $\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%, W_0 = 3$.

Figure 2: Optimal $c$ as a function of $\gamma$. Parameter values: $\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%, W_0 = 3$. 
Figure 3: Optimal $c$ as a function of $k$. Parameter values: $\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%, W_0 = 3$.

3.3 Contingent collateralization

We now generalize the previous example, by allowing for collateral rules that may be contingent on the path of the state variable vector $(W, Z, C)$. We take the class of the admissible collateral rules to include processes with continuous paths that are predictable with respect to the information generated by the state vector $(W, Z, C)$. For each time $t \in [0, T \wedge \tau)$, we define the collateral amount posted by party $i$ as

$$C_t^i = \int_0^t c^i(s, W_s^i, Z_s, C_s^i) dZ_s,$$

and further require $c^i$ to satisfy the following symmetry conditions:

$$\begin{cases} 
C_t^i = -C_t^j \\
c^i(t, W_t^i, Z_t, C_t^i) = -c^j(t, W_t^j, Z_t, C_t^j) = -c^j(t, -W_t^j, Z_t, -C_t^j). 
\end{cases}$$ (3.5)

The optimal solution now entails:
Proposition 3.5. Consider problem (3.3), with \( k > 0 \) and \( C^i \) optimally chosen in the space of admissible contingent collateral rules defined above. Then, the optimal collateralization strategy is independent of wealth, \( c^i_*(t, W^i_t, Z^i_t, C^i_t) = \hat{c}^i(t, Z^i_t, C^i_t) \), and results in a positive collateral fraction on \((0, T \wedge \tau)\) for \( k = 1 \).

So similarly to before, collateral is positive if there is a hedging demand as it mitigates counterparty risk. To obtain some more insights on the form of the optimal collateral rule, we derive numerical results by solving the Hamilton-Jacobi-Bellman (HJB) PDE associated with the optimal stochastic control problem.\(^6\)

Figure 4 shows the optimal collateralization fraction \( \hat{c}^i \) for different times to maturity and for the average value of \((Z, C)\), whereas Figures 5 depicts the surface representing the optimal collateral rule \( \hat{c}^i \) for different configurations of the pair \((Z, C^a)\). The findings show similarities to the deterministic collateral case treated in the previous section. Specifically, optimal collateral levels again are positive due to the considerable hedging demand \((k = 0.9)\) — yet they are lower than "full" collateralization. This comes as no surprise since a similar intuition prevails: The counterparties evaluate scenarios where collateral is paid and where collateral is received differently, and the difference in marginal utilities pushes the levels away from full collateralization. Interestingly, the collateral levels do vary between different states \((Z, C^a)\). A possible explanation for this observation is that unlike the deterministic collateral case, state dependence allows to mitigate the exposure to fluctuations in the terminal wealth due to the collateral flows.

\(^6\)Because of the symmetry requirements (3.5), the HJB for this problem gives rise to a PDE with distributed arguments that needs a suitable extension of standard finite difference methods for boundary value problems.
Figure 4: Optimal collateral fraction for average value of the pair \((Z, C)\) and for different times to maturity. Parameter values: \(\gamma = 0.2, T = 1, \lambda = 5\%, k = 0.9, \sigma_z = 20\%, r = 3\%, \mu = 6\%, \sigma_S = 20\%\).

Figure 5: Optimal collateral fraction \(\hat{c}\) as a function of the collateral posted to date, \(C\), and realization of \(Z\). Parameter values as in figure 4.
4 Discussion

4.1 Re-hypothecation of collateral

In the presentation of the model in sections 2 and 3, we have always assumed that collateral amounts received from a counterparty become part of the trading account and can therefore be invested in the financial market. More generally, collateral could be re-pledged for other purposes, potentially exposing the counterparty to delays in collateral transfers, should the position turn in her favour, and to close-out risk, in case excess collateral has been posted and was used to support trades difficult to unwind. For these reasons collateral is often segregated and re-hypothecation not allowed, meaning that the term \((1 - N_t) dC_t\) would not appear in the budget dynamics (3.1), as collateral would be posted in a separate account managed by a custodian. Our setting can easily accommodate these changes and provide similar results. However, \(ceteris paribus\) the optimal collateral fractions would be uniformly lower than in the case of re-hypothecation, as the benefits from holding collateral would now be limited to mitigating counterparty risk (rather than generating excess returns from investment of additional resources in the financial market), while the costs of posting collateral would be insensitive to segregation in our setting.

4.2 Full collateralization

It is clear from the results of the previous section that full collateralization is in general a suboptimal strategy. The fact that such strong requirement is common in some OTC markets (e.g., in interest rate swaps; see ISDA, 2010b), should be interpreted in the light of frictions affecting the collateralization process in real-world transactions. In
practice the collateralization process is discrete and, depending on the specific asset class or products considered, collateral revisions may be infrequent. In this situation full collateralization may provide a simple way to provide a buffer against gaps in the protection against counterparty risk originating from discrete collateralization.

There is an additional important issue to note, however. The term 'full collateralization' is often a misnomer in OTC transactions, as counterparty risk mitigation is 'full’ to the extent of the replacement cost defined by the CSA.\(^7\) The definition is not unique, and often ambiguous, as there may be cases when liquidators are given the opportunity to choose between a risk-free close-out, such as the one considered in (3.2), and a credit risky close-out, for example estimated from a range of quotes obtained in the market after the default event. Even if the definition were unambiguous, it is clear that collateralization would be full only if the replacement cost were to coincide with the market-value of the OTC contract being considered. This is often not the case, as CSAs would typically specify proxies and models to be used to determine collateral amounts, which have nothing to do with the value of the contract.

5 Conclusion

In this work we have examined OTC transactions subject to bilateral default risk. In our baseline setting agents are exposed to a nontradable source of risk that can be hedged by entering a risk sharing agreement with a counterparty that is credit risky. Counterparty risk can be mitigated by suitably designing a CSA indicating when and how much

\(^7\)A different situation is when the maximum payout from an instrument can be defined and collateral is posted at inception and segregated. This is the case of catastrophe bonds and other securitized products (e.g., Lakdawalla and Zanjani, 2012). Even there, however, collateralization is not entirely full as cash collateral is too expensive and fixed income instruments offering higher yields (but exposed to nonzero credit risk) are often used.
collateral to post at each given point in time during the life of the transaction. We have determined optimal collateral rules over different admissible strategies, therefore providing a microeconomic foundation for collateralization strategies observed in practice. At the same time we have developed a framework that allows one to address several issues related to the delicate features of OTC transactions, such as re-hypothecation and segregation of collateral, the role of funding costs, and the definition of close-out conventions. Our setting shows that funding costs, which for us arise endogenously as the opportunity cost of optimally allocating resources to a trading technology, are an integral part of optimal CSA design. At the same time, we have demonstrated how our framework can be used to provide predictions on the effects of mandating suboptimal collateral rules or of imposing segregation of collateral and banning its re-hypothecation.

References


