Precautionary Insurance Demand with State-Dependent Background Risk

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Abstract

This paper considers a zero-mean background risk that is uncorrelated with insurable losses, but is not necessarily statistically independent. In particular, the size of the background risk can vary in different insurable-loss states. With a background risk only in the loss state, prudence together with risk aversion guarantees an increase in insurance demand. However, with a background risk only in the no-loss state, prudence guarantees that we actually reduce the demand for insurance in the presence of such background risk. Moreover, if we consider two individuals, with one more risk averse than the other, we need to compare the intensities of their precautionary motives, in addition to their measures of risk aversion, before we can determine who buys more insurance coverage in the presence of the state dependent background risk.

Keywords: background risk, insurance demand, precautionary demand, prudence, risk aversion, Ross risk aversion

JEL classification: D81
1 Introduction

Many models have been developed to examine insurance decisions in the presence of other risks. When one of the risks is exogenous and unhedgeable, we typically refer to it as a "background risk" and much research in the past 20 years has examined how the presence of this background risk affects decisions regarding the insurable risk.\footnote{Doherty and Schlesinger (1983) and Mayers and Smith (1983) first examined this issue. A good summary of most of the key results can be found in Gollier (2001).} Most of these studies focus on the case where the background risk is statistically independent from the endogenous risk. To our knowledge, Eeckhoudt and Kimball (1992) are the first to provide conditions on the utility function, such that the presence of an independent background risk makes people act in a more risk-averse manner towards the purchase of insurance. A few papers, including Doherty and Schlesinger (1983) as well as Eeckhoudt and Kimball (1992) also examine cases for which the background risk is statistically dependent on the insurable risk.

In addition to research examining the effects of background risk on an individual's behavior, another line of research has looked at whether or not interpersonal comparisons
of insurance-purchasing behavior will continue to hold in the presence of a background risk. Kihlstrom, Romer and Williams (1981) and Nachman (1982) were the first to show that a more risk-averse individual might not continue to purchase more insurance in the presence of an independent background risk. Ross (1981) looked at a similar question but assumed that the insurable risk and the background risk had a zero correlation, which is weaker than the assumption of independence.

In this paper, we study the effect of a state-dependent, zero-mean background risk on the demand for insurance. In particular, we assume that the conditional mean of the background risk, given any realization of insurable losses, is always zero. Since such a background risk has a zero correlation with insurable loses, there is no possibility of using insurance for purposes of cross hedging, as in Schlesinger and Doherty (1985). However, the fact that the background risk can vary in different loss states leads to a precautionary effect of insurance. Even though the background risk is not cross-hedgeable via insurance, the consumer can adjust wealth levels in different loss states in order to better cope with the background risk. For a prudent consumer, changes in insurance can shift more wealth to states for which the background risk is relatively high, thus mitigating the untoward effects of the background risk.

The paper is similar in spirit to that of Eeckhoudt, Gollier and Schlesinger (1991), who consider localized increases in risk within a model of deductible insurance. Although they consider a change in the distribution of the insurable risk, the localized nature of their risk changes has effects similar to those we obtain in a setting of localized background risks, as we explain in the text.
We first set up a model with localized background risk and compare it to one with an independent background risk. We then show how prudence plays a crucial role in determining whether or not insurance demand increases in the presence of background risk. If the background risk in the different loss states varies only by a size factor (i.e. a scaling effect), we show how changes in relative scaling affect the demand for insurance. We next consider two consumers, with one being more risk averse than the other. We examine conditions under which the more-risk-averse individual will demand more insurance in the presence of the background risk and we show how their comparative levels of prudence play a role in determining their comparative demands for insurance coverage.

2 The Model

We use the simplest model of insurable risk, in which there are only two loss events, "loss" and "no loss," which partition the state space. An insured with an initial wealth level $W$ also has an insurable risk $x$. We assume that the loss $x$ has a value of $L \leq W$ with probability $p$ and it has a value of zero with probability $1 - p$. We assume that the insured is strictly risk-averse with her utility function $u$ satisfying the conditions $u' > 0$ and $u'' < 0$. We also assume that $u$ is at least thrice differentiable. An insurance contract is available which charges a premium $P(\alpha) = (1 + m)\alpha pL$ and pays an indemnity $\alpha L$ in the event of a loss, where $\alpha$ is the rate of coinsurance, and $m \geq 0$ is the so-called "loading factor" for the premium. We require that $(1 + m)p < 1$, since if this was not the case, the insurance premium would be higher than the indemnity that was paid for any
loss and no insurance would be sold. In the absence of any background risk, the insured chooses the optimal amount of insurance to maximize his expected utility as follows:

$$\max \alpha EU \equiv pu(W - P(\alpha) - L + \alpha L) + (1 - p)u(W - P(\alpha)).$$ (1)

The optimal rate of coinsurance $\alpha^*$ satisfies the necessary first-order condition

$$[p(1 - p - mp)L]u'(y_L) - [p(1 - p - mp + m)L]u'(y_N) = 0,$$ (2)

where $y_L \equiv W - P(\alpha) - L + \alpha L$ and $y_N \equiv W - P(\alpha)$ are notations for wealth in the loss and no-loss states respectively. Straightforward calculations show that $EU$ as defined in (1) is concave in $\alpha$, so that second-order conditions for a maximum also hold. For the sake of simplicity, we assume that $\alpha^* > 0$.\(^2\)

If $m = 0$, it follows trivially from (2) that $y_L = y_N$, so that full insurance is optimal, $\alpha^* = 1$. If $m > 0$, it also follows easily from (2) that partial insurance is preferred, $\alpha^* < 1$. These two results are generally attributed to Mossin (1968).

Suppose we now consider another consumer with utility $v(y)$, who is uniformly more risk averse than the consumer with utility $u(y)$. For ease of exposition, we refer to these individuals as $v$ and $u$ respectively. Following Pratt (1964), it follows that for any $m > 0$, the optimal level of insurance for consumer $v$ will be higher than that of consumer $u$, ceteris paribus, i.e. $\alpha^*_v > \alpha^*_u$.

\(^2\)All of the results can easily be extended to allow for $\alpha^* < 0$ or for a corner solution at $\alpha^* = 0$. However, such results do not add any new insights and only complicate the mathematics.
We assume that any background risk for the individual takes the form $\tilde{\theta}_L$ in the
dates of the world for which $x = L$ and the form $\tilde{\theta}_N$ in the states of the world for which
$x = 0$. We also assume that $E\tilde{\theta}_L = E\tilde{\theta}_N = 0$. If $\text{var}(\tilde{\theta}_L) > 0$ and $\text{var}(\tilde{\theta}_N) = 0$, then
we have a case in which background risk only occurs if there has been a loss, and there
is no background risk in the no-loss states. If $\text{var}(\tilde{\theta}_L) = 0$ and $\text{var}(\tilde{\theta}_N) > 0$, then
we have just the opposite: a case in which background risk only occurs if there is no
insurable loss.

Since this setting is a bit too general, we simplify it to allow only for the relative
size of the background risk to vary in the loss states vs. the no-loss states. To this
end, let $\tilde{\varepsilon}$ be a zero-mean random variable with a variance that is positive (i.e. $\tilde{\varepsilon}$ is not
identically zero). We assume that $\tilde{\varepsilon}$ is not directly hedgeable and further assume that
$\tilde{\varepsilon}$ and $\tilde{x}$ have a zero correlation. Thus, one cannot indirectly hedge against the $\tilde{\varepsilon}$-risk
via the purchase of additional insurance against $\tilde{x}$. We let $\beta$ be a scalar $0 \leq \beta \leq 1$, and
define $\tilde{\theta}_L = \beta \tilde{\varepsilon}$ and $\tilde{\theta}_N = (1 - \beta)\tilde{\varepsilon}$. If $\beta = 1$, we have a background risk only in the
loss states. If $\beta = 0$, we have a background risk only in the no-loss states. If $\beta = \frac{1}{2}$,
then $\tilde{\theta}_L = \tilde{\theta}_N = \frac{1}{2} \tilde{\varepsilon}$, which corresponds to the case of an independent background risk.
Obviously, as $\beta$ varies, we can tell other stories. For instance, if $\beta$ is very small, then
we have a background risk in the loss states, but there is still a very small amount
of background risk in the no-loss states.

We next examine the basic results above to see how they are affected by the addition
of a background risk.
3 Background Risk and Insurance Demand

The consumer’s objective in the presence of the background risk now can be written as follows:

$$\max_{\alpha} EU \equiv p E[u(y_L + \beta \tilde{z}) + (1 - p) E[u(y_N + (1 - \beta) \tilde{z})].$$  \hspace{1cm} (3)

Let $y_L^*$ and $y_N^*$ denote the consumer’s optimal wealth levels for the events "loss" and "no loss," without a background risk, i.e. $y_L^* \equiv W - P(\alpha^*) - L + \alpha^* L$ and $y_N^* \equiv W - P(\alpha^*)$, where $\alpha^*$ is the solution to (1).

For the case where $\beta = \frac{1}{2}$ and with $m > 0$, we know from Gollier and Pratt (1986), that $\alpha^*$ always will increase in the presence of the background risk if and only if preferences are "risk vulnerable." Since risk vulnerability is a difficult property to characterize, they also provide us with several sufficient conditions, the most well-know being "standard risk aversion," as defined by Kimball (1993).\footnote{The sufficiency of standard risk aversion was first shown directly by Eeckhoudt and Kimball (1992). Standard risk aversion can be characterized as having a utility function that exhibits both decreasing absolute risk aversion and decreasing absolute prudence.} For the case where insurance is actuarially fair, $m = 0$, it follows easily that we still require $y_L^* = y_N^*$, so that full coverage remains optimal.

Consider now the case where $\beta = 1$. In this case, a background risk manifests itself only when a loss occurs. For example, we might have other ancillary expenses and benefits associated with a loss. We might need to miss a work opportunity in order to take the time to apply for insurance benefits, or relatives might send us money to help us deal with our loss. In this case, standard risk aversion is stronger than necessary.
to guarantee an increase in the level of insurance coverage. Indeed, we do not even require the weaker condition of decreasing absolute risk aversion. Rather, the property of "prudence," \( u'' > 0 \), is both necessary and sufficient to guarantee an increase in insurance.

Since \( EU \) is still concave in \( \alpha \) in the presence of a background risk, we can determine the effect of background risk by evaluating the sign of \( dEU/d\alpha \), evaluated at the optimal level of coinsurance without background risk \( \alpha^* \),

\[
\frac{dEU}{d\alpha}|_{\alpha^*} = \left[ p(1 - p - mp)L\right]u'(y_L^* + \bar{\varepsilon}) - \left[ p(1 - p - mp + m)\right]u'(y_N^*). \quad (4)
\]

Comparing (4) with the first-order condition without background risk (2), it follows easily from Jensen’s inequality that \( dEU/d\alpha > 0 \) if and only if \( Eu'(y_L^* + \bar{\varepsilon}) > u'(y_N^*) \).

From Jensen’s inequality, this in turn will follow for any arbitrary zero-mean risk \( \bar{\varepsilon} \) and for any arbitrary starting wealth \( W \) if and only if \( u' \) is convex in wealth, i.e. consumer \( u \) is prudent. Since we assume that \( u \) is thrice differentiable, this is equivalent to \( u'''' > 0 \) everywhere, with \( u'''' > 0 \) on a set of positive probability measure on the subset of loss states. Thus, a necessary and sufficient condition for insurance to increase in the presence of background risk only in the loss states, is that the consumer is prudent. Note that this result will hold even with fair insurance, \( m = 0 \). That is, with fair insurance, the consumer will buy more than full coverage, \( \alpha^* > 1 \).

The intuition for this result follows from Kimball (1990), who analyzes the demand for precautionary savings. Here we do not have two periods, but rather a partition of
the states of nature into those with and without the occurrence of the insurable loss. Equation (2) shows how insurance is chosen to balance the marginal net utility benefit in the loss states from an increase in $\alpha$, with the marginal net utility cost in the no-loss states, stemming from the associated higher premium. When we now make wealth in the loss states risky by adding a background risk, the prudent consumer can mitigate the loss of utility by shifting some more wealth to the loss states.

The logic is the same as for precautionary savings, in which a prudent investor shifts more wealth to a later period (i.e. saves more) to better cope with the introduction of a risk in future labor income. In the present case, the consumer uses the insurance contract not only to directly hedge against the loss $\bar{x}$, but also in a precautionary sense to shift wealth into the states with the background risk. For example, if $m = 0$, the consumer not only buys enough insurance to fully hedge the $\bar{x}$-risk, but buys a bit more. By purchasing this excess insurance, the consumer shifts a little bit more wealth from the no-loss states into the loss states, which lowers the "pain" caused by the background risk $\bar{\varepsilon}$, where "pain" here is measured as the loss of utility.4

If $\beta = 0$, background risk occurs only in the no-loss states. For instance, suppose that initial wealth is composed of cash in the amount of $W - L$, plus an asset with a monetary value of $L + \bar{\varepsilon}$. If the asset is stolen, we have only a wealth of $W - L$. Since a realization of $\bar{\varepsilon}$ cannot be observed, insurance can only be based on the mean value $L$. As a result, with no insurance wealth in the loss states is $W - L$, while wealth in the

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4 This notion of increasing wealth to stem the pain follows from Eeckhoudt and Schlesinger (2006), who use the utility premium of Friedman and Savage (1948) as a measure of "pain."
The purchase of insurance in this setting can be compared to the case of no background risk by once again examining the sign of $dEU/d\alpha$, evaluated at the optimal level of coinsurance without background risk $\alpha^*$. 

$$\frac{dEU}{d\alpha}|_{\alpha^*} = [p(1 - p - mp)L]u'(y^*_L) - [p(1 - p - mp + m)L]E[u'(y^*_N + \bar{e})].$$

Unlike for the case where $\beta = 1$, in this case a prudent consumer actually would purchase less insurance. Indeed, when $\beta = 0$, insurance would necessarily increase if and only if $u'$ is concave, i.e. preferences are imprudent with $u'' < 0$. In the case where insurance is fair, $m = 0$, the prudent consumer (with $u'' > 0$) purchases less than full coverage. Again the reasoning is precautionary. By choosing $\alpha^* < 1$, the consumer can save on some of her premium expenditure, thus effectively shifting some of her wealth to the no-loss states so as to mitigate the "pain" from the background risk.

Of course for the well known case where $\beta = \frac{1}{2}$, we need the precautionary effect in the loss states (to increase insurance coverage) to outweigh the precautionary effect in the no-loss states (to decrease insurance coverage), which essentially requires that prudence be decreasing in wealth.

If $\beta$ is close to zero or close to one, we might expect that behavior is similar to that where the background risk only occurs in the loss states or only in the no-loss states. But, of course, how "close" would $\beta$ need to be? As it turns out, assuming a prudent consumer, the optimal level of insurance coverage for a fixed loading $m$ is
strictly increasing in $\beta$: as $\beta$ increases so does the optimal level of insurance $\alpha^*$. This follows from simply differentiating the first-order condition for the general case in (3):

$$
\frac{\partial^2 EU}{\partial \alpha \partial \beta}|_{\alpha^*} = [p(1 - p - mp)L]E[u''(y_L^* + \beta \bar{\varepsilon})\bar{\varepsilon}] + [p(1 - p - mp + m)L]E[u''(y_N^* + (1 - \beta)\bar{\varepsilon})\bar{\varepsilon}].
$$

(6)

It turns out that both $E[u''(y_L^* + \beta \bar{\varepsilon})\bar{\varepsilon}]$ and $E[u''(y_N^* + (1 - \beta)\bar{\varepsilon})\bar{\varepsilon}]$ are positive, as we show in an appendix. Hence, the derivative in (6) is positive and thus a higher $\beta$ begets a higher level of insurance.

The above result also implies that there exists some critical level $\hat{\beta} \in (0, 1)$ such that the optimal level of insurance is the same with background risk as without background risk for this critical value of $\beta$. This follows trivially from the fact that the optimal level of insurance coverage is strictly increasing in $\beta$, together with the fact that the optimal levels of coverage with background risk are lower than the no-background-risk optimum $\alpha^*$ for the case with $\beta = 0$ and higher than the no-background-risk optimum $\alpha^*$ for this case with $\beta = 1$. It thus also follows that the optimal level of insurance in the presence of the background risk is always higher [lower] than the no-background-risk optimum $\alpha^*$ whenever $\beta > \hat{\beta}$ [$\beta < \hat{\beta}$].

4 Interpersonal Comparisons of Insurance Demand

Kihlstrom, Romer and Williams (1981) showed that if consumer $v$ is more risk averse than consumer $u$, so that consumer $v$ would purchase more insurance when $m > 0$ in the absence of any background risk, it might not follow that consumer $v$ will continue
purchase more insurance than consumer $u$ in the presence of an independent background risk. However, consumer $v$ will continue to purchase more insurance than consumer $u$ in the presence of the background risk if either $v$ or $u$ exhibits non-increasing absolute risk aversion.\textsuperscript{5} In our model, this is the case where $\beta = \frac{1}{2}$.

Not surprisingly, non-increasing absolute risk aversion by one of the two consumers is no longer sufficient for $v$ to purchase more insurance in cases where $\beta \neq \frac{1}{2}$. We illustrate this for the two cases where $\beta = 1$ and $\beta = 0$ by way of examples. In both examples below, we set $W = 1$, $L = 0.5$, $p = 0.01$ and the random variable $\tilde{e}$ takes on the values $\pm 0.3$, each with a 50 percent chance of occurrence.

\textbf{EXAMPLE 1.} Let $\beta = 1$, so that a background risk occurs only in the loss states. We define $u(w) = -(w + 160)^{-0.5}$ and $v(w) = -(w - 100)^2$ for $0 < w < 100$. Thus absolute risk aversion is easily calculated as $A_u(w) = 1.5(w + 160)^{-1} < 0.0094$ for all $w$ and $A_v(w) = (100 - w)^{-1} > 0.010$ for all $w$. Thus $v$ is everywhere more risk averse than $u$. Moreover, risk aversion is decreasing in wealth for consumer $u$. The optimal level of coinsurance is shown as a function of the loading factor $m$ in Figure 1. In this Figure we see that the demand for consumer $v$ is sometimes higher and sometimes lower than for person $u$, even though we know that for each $m$ consumer $v$ would demand more insurance than consumer $u$ in the absence of any background risk. This example also illustrates how for the prudent consumer $v$, more than full coverage is demanded at

\textsuperscript{5}This is a sufficient condition. Nachman (1982) introduces a more general condition: if $A_u(w)$ denotes the measure of absolute risk aversion as a function of wealth for consumer $u$, and $A_v(w)$ denotes the same measure for $v$, then $v$ will continue to purchase more insurance than $u$ in the presence of the independent background risk if there exists a non-increasing function $A(w)$, such that $A_v(w) \geq A(w) \geq A_u(w)$ for all $w$. 

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a fair price, due to a precautionary effect. On the other hand, since $u'' = 0$ everywhere, there is no precautionary effect for consumer $u$.

![Insurance Demand at Different Prices (Example 1)](image)

**FIGURE 1, $\beta = 1$**

**EXAMPLE 2.** Let $\beta = 0$, so that a background risk occurs only in the no-loss states. We define $u(w) = \ln(w + 4)$ and $v(w) = \ln(w + 2)$. Thus absolute risk aversion is easily calculated as $A_u(w) = (w + 4)^{-1}$ and $A_v(w) = (w + 2)^{-1}$. Hence, consumer $v$ is once again everywhere more risk averse than consumer $u$. In this example, risk aversion for both consumers is decreasing in wealth. Since both consumers are prudent, we see in Figure 2 that only partial coverage is demanded, even at a fair price with
This is due to the precautionary effect when $\beta = 0$. But we also see in Figure 2 how the demand for consumer $v$ is not everywhere higher than for the less-risk-averse consumer $u$.

![Insurance Demand at Different Prices (Example2)](image)

**FIGURE 2, $\beta = 0$**

Since prudence played a role in determining the demand for insurance, it is not too surprising that comparative measures of prudence between consumers $v$ and $u$ play a role in determining the effects of background risk on their relative insurance demand. This is easily shown using Kimball’s (1990) "precautionary premium." For any thrice differentiable utility $u$ and any zero-mean risk $\tilde{\varepsilon}$, define the precautionary premium
\( \varphi_u(y, \tilde{\varepsilon}) \) implicitly via \( u(y - \varphi_u(y, \tilde{\varepsilon})) \equiv Eu(y + \tilde{\varepsilon}) \). Kimball shows that, for any arbitrary wealth \( y \) and arbitrary risk \( \tilde{\varepsilon} \), \( \varphi_v(y, \tilde{\varepsilon}) > \varphi_u(y, \tilde{\varepsilon}) \) if and only if consumer \( v \) is more prudent than consumer \( u \).

We consider first the case where the background risk manifests itself only in the loss state, \( \beta = 1 \).

**Proposition 1** Let \( \beta = 1 \). Given any \( m \geq 0 \), consumer \( v \) will always purchase more insurance than consumer \( u \) for every arbitrary \( W, L \) and \( \tilde{\varepsilon} \), if and only if consumer \( v \) is both more risk averse and more prudent than consumer \( u \).

**Proof.** The first-order condition for consumer \( u \), (4), can be re-written as

\[
\frac{dEU}{d\alpha}|_{\alpha^*} = [p(1 - p - mp)L]u'(y^*_L - \varphi_u) - [p(1 - p - mp + m)L]u'(y^*_N) = 0, \quad (7)
\]

where \( \alpha^* \) denotes the optimal level of insurance coverage for consumer \( u \). Since utility is unique only up to an affine transformation, assume without losing generality that \( v'(y^*_N) = u'(y^*_N) \). \(^6\) Now

\[
\frac{dEV}{d\alpha}|_{\alpha^*} = [p(1 - p - mp)L]v'(y^*_L - \varphi_v) - [p(1 - p - mp + m)L]u'(y^*_N)
> [p(1 - p - mp)L]v'(y^*_L - \varphi_u) - [p(1 - p - mp + m)L]u'(y^*_N) \quad (8)
> [p(1 - p - mp)L]u'(y^*_L - \varphi_u) - [p(1 - p - mp + m)L]u'(y^*_N)
\]

The first inequality in (8) stems for the fact that \( \varphi_v > \varphi_u \) and that \( v' \) is decreasing. The

\(^6\)In other words, simply multiply the original utility \( v \) by the positive constant \( u'(y^*_N)/v'(y^*_N) \), which represents the same risk preferences as the original utility \( v \).
second inequality stems from Theorem 1 in Pratt (1964), together with the assumption that \( v'(y^*_N) = u'(y^*_N) \). Comparing (8) with (7), it follows that \( \frac{dEV}{d\alpha}|_{\alpha^*} > 0 \), so that the optimal level of insurance coverage for consumer \( v \) must be higher than \( \alpha^* \). This proves sufficiency.

The necessity of higher risk aversion for consumer \( v \) is seen most readily by letting the epsilon risk get very small. Likewise, necessity for higher prudence follows by letting the size of the loss, \( L \), approach zero. ■

The two scenarios described for necessity the proof above help with the intuition of what is involved. Obviously, if the background risk is negligible, we are reduced to the well known result that a propensity for more insurance is equivalent to higher degree of risk aversion. Similarly, if the loss size \( L \) is negligible, then a "loss" is only important due to the \( \bar{\varepsilon} \) risk that accompanies it. In this case, insurance has only a precautionary value, as described by Kimball (1990).

**Proposition 2** Let \( \beta = 0 \). Given any \( m \geq 0 \), consumer \( v \) will always purchase more insurance than consumer \( u \) for every arbitrary \( W, L \) and \( \bar{\varepsilon} \), if and only if \( v \) is both more risk averse than consumer \( u \), but less prudent than consumer \( u \).

The proof is almost identical to Proposition 1 and is omitted. It is important to note that in Proposition 2 we require that \( v \) be less prudent than \( u \). The intuition here should be obvious from the previous section. For the consumer who is prudent, the prudence effect is to reduce the level of insurance. This allows the consumer to save some of the premium as a precaution against the \( \bar{\varepsilon} \) risk, which occurs only in the no-loss
states. If consumer $v$ is less prudent, she has a smaller precautionary effect, to reinforce her stronger effect of higher risk aversion. If consumer $v$ is both more risk averse and more prudent than consumer $u$, then we cannot say who will purchase more insurance, \emph{a priori.}

As we remarked earlier, the case where $\beta = \frac{1}{2}$ corresponds to an independent background risk and was essentially examined by Kihlstrom, Romer and Williams (1981) and Nachman (1982). However, a similar framework was also examined by Ross (1981). Ross examined the case where the risky wealth always had the same conditional mean, regardless of the realized value of the background risk. In particular, for our setting, if random wealth is denoted as $\bar{y}$ and the background risk is $\bar{\theta}$, then $E(\bar{y} \mid \bar{\theta} = c) = E\bar{y}$ (the unconditional mean) for every $c$ in the support of the random variable $\bar{\theta}$. This is clearly true when $\beta = \frac{1}{2}$, but clearly not true for $\beta \neq \frac{1}{2}$. Since $\bar{\theta}$ is defined conditionally as either $\beta \bar{\varepsilon}$ or $(1 - \beta)\bar{\varepsilon}$, if $\beta = 0$, then any value of $\bar{\theta} \neq 0$ will signal that we are in the states of the world in which no loss has occurred. So, for example, if $\bar{\varepsilon}$ has a continuous distribution, then a value of $\bar{\theta} = 0$ will signal that a loss has occurred.\footnote{This follows since, with a continuous distribution, the probability that $\bar{\varepsilon} = 0$ is itself zero. So that observing $\bar{\theta} = 0$ guarantees that we must be observing $\beta \bar{\varepsilon}$ (since $\beta = 0$). Thus, we must be in a state of the world in which a loss has occurred.} Thus, we see in this case that $E(\bar{y} \mid \bar{\theta} = 0) = y_L$, whereas $E(\bar{y} \mid \bar{\theta} \neq 0) = y_N$.

In terms of risk aversion, Ross proposed a stronger measure of risk aversion for which $v$ is more risk averse than $u$ in the strong sense of Ross if $\exists \lambda > 0$, such that

$$\frac{v''(x)}{w''(x)} \geq \lambda \geq \frac{v'(y)}{w'(y)} \text{ for all } x \text{ and all } y.$$  \hspace{1cm} (9)
Clearly this definition is stronger than the usual (Arrow-Pratt) definition of more risk averse. Ross also shows how the inequality in (9) above, implies the existence of a real-valued function $G$, with $G' < 0$ and $G'' < 0$, such that $v(y) = u(y) + G(y)$ for all $y$.

Turning to insurance, let $\alpha^*$ denote the optimal level of insurance for consumer $u$. It follows easily that for $\beta = \frac{1}{2}$ and $m > 0$, for consumer $v$ we have

$$dEV_{\alpha^*} = [p(1 - p - mp)L]E[u'(y_L^* + \frac{1}{2}\varepsilon)] + G'(y_L^* + \frac{1}{2}\varepsilon)$$

$$- [p(1 - p - mp + m)L]E[u'(y_N^* + \frac{1}{2}\varepsilon)] + G'(y_N^* + \frac{1}{2}\varepsilon)]$$

$$= [p(1 - p - mp)L]EG'(y_L^* + \frac{1}{2}\varepsilon) - [p(1 - p - mp + m)L]EG'(y_N^* + \frac{1}{2}\varepsilon) > 0. \tag{10}$$

The last inequality follows since $y_L^* < y_N^*$ and $G'$ is negative and decreasing. Thus, for every $\varepsilon$, $|G'(y_L^* + \frac{1}{2}\varepsilon)| < |G'(y_N^* + \frac{1}{2}\varepsilon)|$, i.e. the first term is less negative than the second term.

Since $dEV_{\alpha^*}$ is continuous in $\beta$, the strict inequality in (10) must hold for $\beta$ close enough to $\frac{1}{2}$. However, in general, for $\beta \neq \frac{1}{2}$, the above analysis need not hold. This can be seen in our two previous examples, with $\beta = 1$ and $\beta = 0$.

In Example 1, with $\beta = 1$, straightforward calculations show that $\frac{v''(w)}{v'(w)} = \frac{8}{3}(w + 150)^{\frac{5}{2}} \geq 863,510$, whereas $\frac{v'(w)}{v''(w)} = 4(100 - w)(w + 160)^{\frac{3}{2}} \leq 809,540$, for all $w$ such that $0 \leq w \leq 100$. Thus, consumer $v$ is more risk averse than consumer $u$ in the strong sense of Ross over all relevant wealth values. Yet we see in Example 1 that consumer $v$ sometimes buys more insurance and sometimes buys less insurance than consumer $u$.

Similarly, in Example 2, one can show that $\frac{v''(w)}{v'(w)} = (\frac{w+4}{w+2})^2 \geq \frac{9}{4}$, whereas $\frac{v'(w)}{v''(w)} = \frac{w+4}{w+2} \leq 2$, for all $w$ in the relevant wealth range, which here is $0 \leq w \leq 2$. Thus, we
once again have consumer $v$ is more risk averse than consumer $u$ in the strong sense of Ross. Yet, as in Example 1, consumer $v$ sometimes buys more insurance and sometimes buys less insurance than consumer $u$.

5 Concluding Remarks

This paper considered a zero-mean background risk that is uncorrelated with insurable losses, but is not necessarily statistically independent. We studied the effect of such a background risk on the demand for insurance. If there is substantially more background risk in the states of the world with an insurable loss, the effect will be to increase the demand for insurance. In the case of a fair premium, the consumer will demand more than full insurance, contrary to Mossin’s Theorem (1968). The rationale for this extra demand is a precautionary motive for insurance. Although the extra insurance in no way hedges the background risk, the extra wealth is on hand for precautionary purposes, to help the individual in the event that the realized background risk turns out to be negative. This is identical to the motive for precautionary savings against future income risk, as described by Kimball (1990).

If the background risk is substantially larger in states where no insurable loss occurs, then the precautionary motive is to purchase less insurance. This is to reduce the premium and thus have more money on hand to handle the potential adverse effects of the background risk in these no-loss states. The exact extent of what we mean by background risk being "substantially larger" in one set of loss states is made more precise in the text.
The above reasoning shows that both risk aversion and prudence provide us with information about total insurance demand. Risk aversion is important in gauging the hedging demand (i.e. the usual reason for insurance purchases), whereas prudence gauges the precautionary demand. Similarly, in comparing two individuals facing identical risky choices, both comparative risk aversion and comparative prudence are required. Risk aversion alone, even in the stronger sense of Ross (1981), is not sufficient to guarantee that the more risk averse individual necessarily buys more insurance.

Such state-dependent types of background risk would seem to make sense in many risk-taking situations. Our modeling here is not normative and we make no claim about values of $\beta$ that might be relevant. If a reader thinks that $\beta$ close to one makes sense in one situation, but $\beta$ closer to one-half seems more reasonable in another setting, so be it. Our point is simply that, for any given weighting of such background risk in loss vs. no-loss states of the world, precautionary effects should not be ignored.

Obviously the model set up in this paper is rather simple. More realistic loss distributions and more intricate state-dependencies for the background risk are bound to make the analysis very complicated. Still, we hope that some of the issues addressed here help with setting the "building blocks" for more complicated scenarios.

6 Appendix

We show here that $E[u''(y_L + \beta \bar{\varepsilon})\bar{\varepsilon}] > 0$. The proof for $E[u''(y_N + (1 - \beta)\bar{\varepsilon})\bar{\varepsilon}] > 0$ is essentially the same, so that $\partial^2 EU/\partial \alpha \partial \beta$ in (6) is positive as claimed. Let $F$ denote the distribution function for $\bar{\varepsilon}$. Assuming that we have prudence, $u''' > 0$, then for any
\( \beta > 0 \) we have
\[
E[u''(y^*_L + \beta \epsilon \bar{\epsilon})] = \int_{-\infty}^{+\infty} u''(y^*_L + \beta \epsilon) \epsilon \ dF(\epsilon) \\
= \int_{-\infty}^{0} u''(y^*_L + \beta \epsilon) \epsilon \ dF(\epsilon) + \int_{0}^{+\infty} u''(y^*_L + \beta \epsilon) \epsilon \ dF(\epsilon) \\
> \int_{-\infty}^{0} u''(y^*_L) \epsilon dF(\epsilon) + \int_{0}^{+\infty} u''(y^*_L) \epsilon dF(\epsilon) = u''(y^*_L) \int_{-\infty}^{+\infty} \epsilon dF(\epsilon) = 0.
\]

We should point out that the inequality above reverts to an equality when \( \beta = 0 \). This is simply due to the fact that risk aversion is a second-order effect, so that the first infinitesimal of \( \epsilon \) risk has no effect of decision making. But in this case, \( 1 - \beta > 0 \), so that a strict inequality will hold for \( E[u''(y^*_N + (1 - \beta) \epsilon \bar{\epsilon})] > 0 \) and \( \partial^2 EU/\partial \alpha \partial \beta \) in (6) is thus strictly positive.

References


