Egalitarian Equivalent Capital Allocation*

By

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Abstract

We apply Moulin’s notion of egalitarian equivalent cost sharing of a public good to the problem of insurance capitalization and capital allocation where the liability portfolio is fixed. We show that this approach yields overall capitalization and cost allocations that are Pareto efficient, individually rational, and, unlike other mechanisms, stable in the sense of adhering to cost monotonicity.

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1. Introduction

Mainstream risk capital allocation methods in financial institutions are grounded in the concept of the marginal cost of risk. Specifically, capital is allocated to each risk in a portfolio based on how much capital is consumed when that risk is expanded at the margin. Such approaches have obvious merit in the context of portfolio optimization, where correct pricing of marginal units of exposure is essential.

Other applications of capital allocation, however, may require fundamentally different methods. One such application is the case where capital must be allocated but the portfolio of risk is fixed. This can occur in insurance markets when a closed block of insurance business is reinsured, or when a runoff company is capitalized or recapitalized.¹ In such cases, allocating capital based on how it is consumed when risk is expanded at the margin is no longer economically relevant. Instead of devising allocation rules to prices which guarantee that the right amount of risk is taken conditional on capitalization, as existing methods are designed to do, we must devise rules to guarantee equitable treatment of participants in the course of choosing optimal capitalization.

Existing methodologies, which allocate the total cost of capital based on how the marginal unit is consumed, can introduce a wedge between individual and collective interests. In particular, individual policyholders which are intensive consumers of the marginal capital unit at the social optimum may be better off with lower levels of capitalization when the total capital cost allocation is being keyed to the consumption of the marginal unit. This relates closely to the notion of stability in allocations: If allocations are unstable with respect to small perturbations, then small changes in risk, capitalization, or in risk

¹ A particularly vivid example of a runoff capitalization is provided by Equitas—which is in the process of discharging the liabilities of multiple Lloyd’s syndicates following the market restructuring of 1993.
measure thresholds can produce large swings in policyholder welfare—which can cause individual policyholders to disagree on the optimal level of capitalization.

This paper is concerned with the latter problem. Specifically, we study how capital cost sharing rules can be designed to guarantee Pareto optimal capitalization acceptable to the various policyholders whose exposures make up the portfolio. We resolve the problem of instability by appealing to the concept of cost monotonicity used in the economic theory of public goods—where cost sharing rules are restricted to produce allocations in a way that no agent will object to the introduction of an improvement to the cost technology.

**Background and Motivation**

Many capital allocation methods essentially boil down to the gradient of a risk measure. Examples include Myers and Read (2001), Denault (2001), Tsanakas and Barnett (2003), Tasche (2004), Kalkbrener (2005), and Powers (2007). Economic justification for the gradient method can be recovered in profit maximization problems where the risk measure serves to constrain risk taking (e.g., Zanjani, 2002; Meyers, 2003; Stoughton and Zechn, 2007). In the latter papers, the gradient of the risk measure accurately reflects the marginal cost of risk, so allocating capital according to the risk measure gradient is consistent with marginal cost pricing.

Consistency with marginal cost seems desirable, but it is important to understand that such consistency does not grant universal application of the allocation method. Merton and Perold (1993) proved that risk capital allocation would generally fail to provide accurate pricing of inframarginal or supramarginal changes to a risk portfolio. Hence, allocating capital according to the gradient method can yield accurate pricing of marginal changes to a risk portfolio, but no more. Applications where the risk portfolio is fixed may require a different approach. Rather than asking how much risk to take, given fixed capital, sometimes one is confronted with the question of how much capital to hold, given fixed risk.

The latter problem fits squarely within the public goods literature, and in particular those papers concerned with cost-sharing mechanisms for providing the optimal amount of the public good. In the case of the insurance company, the public good is capital. The policyholders of the company are the consumers, who all enjoy access to the protection afforded by the capital of the firm.

The classic solution to the problem of public good cost sharing is provided by Lindahl (1958), whose basic idea was to derive “personalized prices” that each consumer could pay for the good. These prices were based on each consumer’s marginal utility associated with the public good at the optimum. This idea was subsequently refined and extended by Foley (1970), Kaneko (1977), and Mas-Colell and Silvestre (1989).

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2 Bauer and Zanjani (2013a) provide a review of gradient methods as well as alternative approaches to capital allocation.

3 Although this paper is concerned with an economic approach to capital allocation, it should also be acknowledged the economic approach—in the sense of taking profit or welfare maximization as the guiding objective—is by no means the only approach to capital allocation. Examples of optimization approaches with different objectives can be found in Dhaene, Goovaerts, and Kaas (2003), Laeven and Goovaerts (2004), and Dhaene, Tsanakas, Valdez, and Vanduffel (2012).
who established, among other things, various conditions to guarantee that the solution was Pareto optimal and part of the core.

Cost-sharing based on valuation of the marginal unit, however, can lead to unappealing and unstable outcomes. In particular, one can construct examples, as we do in Section 3.1, where small changes in the level of public good production yield large changes in the cost allocation. In other words, while the Lindahl solution yields a Pareto optimal outcome, the mechanics of the cost sharing can lead some consumers to prefer super-optimal or sub-optimal production levels in cases where a deviation significantly alters the cost allocation. A similar technical problem surfaces in the capital allocation literature. Previous research has recognized the possibility that allocations might not be stable (e.g., Myers and Read, 2001; Zanjani, 2010) to small perturbations of the portfolio or capitalization level.

In the context of the general public goods literature, Moulin (1987) introduced an additional restriction on cost sharing dubbed cost monotonicity aimed at this problem. He argued that a cost sharing mechanism should satisfy the property that all consumers would benefit from a technological improvement in the cost function. This additional restriction, in conjunction with some other conditions on preferences and technology, leads to a unique solution: Specifically, the cost sharing mechanism ends up adhering to what Moulin dubbed egalitarian equivalent cost sharing of a public good. This sharing rule allocates cost so that each consumer’s resulting utility matches her egalitarian equivalent utility level. The consumer’s egalitarian equivalent utility level is the utility she would have received at the maximum level of free public good output that results in a feasible utility distribution.

We adapt this idea to the context of the capital allocation problem in insurance, showing that the egalitarian equivalent approach to cost sharing yields stable capital allocations. The capitalization solution, moreover, is Pareto optimal, and participation in the scheme is individually rational.

The rest of this paper is organized as follows. Section II sets up the insurance capital allocation problem, defines the notion of egalitarian equivalent capital allocations, and shows that the resulting allocations are Pareto optimal, cost monotonic, and individually rational. Section III starts with simple numerical examples demonstrating the stability of the egalitarian equivalent allocations, even in situations where traditional methods yield unstable allocations. It proceeds to analyse allocations based on a set of simulated loss data obtained from a major catastrophe reinsurer. Section IV concludes.

2. Insurance Capitalization and Cost Allocation

We consider a set of $N$ consumers. Each consumer is endowed with $w^i$ and exposed to a random loss variable $L^i$. Each consumer has a contract with the same insurance company promising some non-

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4 To elaborate, feasible utility distributions are ones that can be achieved when the public good is being paid for in full. To find an egalitarian equivalent utility distribution, we go through a process of finding the highest possible level of free public good production that is associated with a feasible utility distribution. Thus, the utility distribution could be achieved in two ways: 1) by providing a free amount of the public good, where every consumer pays a cost share of zero, or 2) by providing a larger amount of the public good, that is fully paid for with cost shares that are allocated to each of the consumers.
negative level of indemnification\(^5\) in the event of loss, denoted \(I^i\). The indemnification level could be full or partial, but is assumed to be less than or equal to the amount of the loss. The recovery from the insurance company may turn out to be less than promised. The company has non-negative assets \(a\) which could be less than total claims, so the consumer’s recovery is:

\[
R^i = \min \left[ I^i, \frac{a}{\sum_{j=1}^{N} I_j^i} \right].
\]

The premium paid by the consumer is denoted by \(p^i\), and we require premiums to cover costs associated with capitalizing the firm. Total costs are assumed to consist of actuarial costs plus a frictional cost \(c(a)\), so that in aggregate:

\[
\sum_{i=1}^{N} p^i = \sum_{i=1}^{N} \mathbb{E}R^i + c(a).
\]

Consumer utility is determined by von Neumann-Morgenstern expected utility, which we will take to be continuous with risk aversion:

\[
\mathbb{E}u^i (w^i - p^i - L^i + R^i).
\]

The premium paid by the consumer, \(p^i\), can be decomposed further into the actuarial loss and an amount to cover the frictional costs of assets:

\[
p^i = \mathbb{E}R^i + z^i,
\]

so that we may write utility as a function of the asset level (the public good) and a cost share

\[
V^i(a, z^i) = \mathbb{E}u^i (w^i - L^i + R^i - \mathbb{E}R^i - z^i).
\]

The properties of expected utility imply that \(V^i(a, z^i)\) is weakly increasing in its first argument and strictly decreasing in its second argument.

We impose the restriction that the cost allocations pay for the (frictional) cost of public good production:

\[
\sum_i z^i = c(a), \quad \text{(1)}
\]

where we take the frictional cost function to be increasing and continuous.\(^6\)

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\(^5\) Note that the contracted indemnity here is taken as a given. For analysis of the optimal level of indemnity, see Zanjani (2010) and Bauer and Zanjani (2016).

\(^6\) We have expressed frictional costs as a function of assets rather than capital, but note that this form is flexible enough to capture frictional capital costs. For example, consider: \(c(a) = \tilde{c}(a - \mathbb{E}R^i)\), where \(\tilde{c}\) is a continuous and increasing function and capital is the difference between assets and expected liabilities \((a - \mathbb{E}R^i)\). Notice that capital is a continuous and increasing function of assets, so that \(c(a)\) will inherit continuity and monotonicity as well.
We write the set of feasible allocations as:

$$\Omega = \{(a, z^1, ..., z^N) \mid a \geq 0, \sum_i z^i = c(a)\}.$$ 

**Asset Level Selection and Cost Sharing Mechanisms**

Given a cost function and a set of $N$ consumers, a mechanism $M$ assigns a level of public good production and a set of cost shares to the consumers satisfying (1):

$$M(c, V^1, ..., V^N) = (a_M(c), z^1_M(c), ..., z^N_M(c)),$$

where $a_M(c)$ is the level of public good production assigned by mechanism $M$ and $z^i_M(c)$ is the cost share assigned to consumer $i$.\(^7\) We are interested here in two key properties of a mechanism:

1. **Pareto optimality** – Does the mechanism select a Pareto optimal allocation for every cost function?

2. **Cost Monotonicity** – A mechanism satisfies cost monotonicity if, for any two cost functions $c_1$ and $c_2$ we have:

   $$c_1(a) \leq c_2(a) \ \forall a \geq 0 \ \Rightarrow V^i(a_M(c_1), z^i_M(c_1)) \geq V^i(a_M(c_2), z^i_M(c_2)) \ \forall i \in \{1, ..., N\}.$$

These properties are less than typically required in the public good literature on cost sharing (e.g., Kaneko, 1977; Mas-Colell and Silvestre, 1989), which usually also requires allocations to be in the “core” (as described by Foley, 1970) of allocations that can’t be improved upon by coalitions of consumers. The core concept is natural in cases where the public good is non-rival, but in the insurance case all consumers are rival claimants on the same assets. Contemplating sub-coalitions of consumers in this case would thus involve alteration of preferences over the public good, as removal of potential claimants affects prospective consumption by the remaining consumers. Consistent with our motivating examples, we restrict our attention to the case where consumers cannot leave the company and thus do not require the allocation to be coalition-proof.

Cost monotonicity was introduced by Moulin (1987), motivated by requiring any cost sharing mechanism to allocate responsibility in such a manner that “no agent will oppose the implementation of a technological advance.” As will become clear in the example of the next section, this requirement is intimately related to the notion of stability in capital allocations: If a mechanism has a tendency to produce allocations that are unstable with respect to small changes in capitalization, it is not likely to be cost monotonic.

\(^7\) Note that the assignments will generally depend on the utility functions, so one could write the assignments as functions of the utility functions in addition to the cost function. At the expense of precision, we choose the simpler presentation to avoid notational tedium and also to underscore our interest in cost monotonicity, which will involves varying the cost function while holding the utility functions fixed.
Egalitarian Equivalent Capital Cost Allocation

Moulin (1987) also introduced the mechanism of egalitarian equivalent cost allocation, an approach he showed to be consistent with cost monotonicity. His idea was to allocate cost responsibility so that the resulting distribution of utility would match the distribution associated with the egalitarian equivalent level of public good production—which he defined as the highest possible level of the public good that, if it were provided for free to consumers, would yield a feasible utility distribution.

In our case, the egalitarian equivalent level of assets $a^*$ is the highest level of assets that would yield a feasible utility distribution, if the policyholders did not have to pay for the frictional costs associated with those assets:

$$a^* = \sup \{ \tilde{a} \geq 0 \mid \exists (a, z^1, ..., z^N) \in \Omega: V^i(\tilde{a}, 0) \leq V^i(a, z^i) \ \forall i \in \{1, ..., N\} \}.$$  

Moreover, given an egalitarian equivalent level of assets $a^*$, we call any feasible allocation $(\tilde{a}, \tilde{z}^1, ..., \tilde{z}^N)$ satisfying:

$$V^i(a^*, 0) = V^i(\tilde{a}, \tilde{z}^i) \ \forall i \in \{1, ..., N\},$$

an egalitarian equivalent allocation.

The following theorems establish existence, individual rationality, Pareto efficiency, and cost monotonicity of egalitarian equivalent allocations. They are essentially slight modifications of portions of Moulin’s results, adapted to the problem at hand and in particular sidestepping the issue of the core property.

**Theorem 1** Suppose the loss distributions are bounded and nontrivial and the cost function is strictly increasing and weakly convex. The egalitarian equivalent level of public good production $a^*$ is finite and any egalitarian-equivalent allocation $(\tilde{a}, \tilde{z}^1, ..., \tilde{z}^N)$ satisfies:

$$V^i(a^*, 0) = V^i(\tilde{a}, \tilde{z}^i) \ \forall i \in \{1, ..., N\}.$$  

**Proof:** See Appendix.

Note that a consequence of Theorem 1 is that egalitarian equivalent allocations satisfy individual rationality. The egalitarian equivalent level of assets must be nonnegative, so any egalitarian equivalent allocation at least weakly dominates the zero allocation which is the relevant one for assessing individual rationality.

$$V^i(a_m, z^i_m) \geq V^i(0,0) \ \forall i \in \{1, ..., N\}.$$  

**Theorem 2** An egalitarian equivalent allocation is Pareto efficient.

**Proof:** See Appendix.
Theorem 3 - An egalitarian equivalent mechanism is cost monotonic.

Proof: See Appendix.

3. Numerical Examples

We start in Section 3.1 by considering various examples involving consumers with binary loss distributions, exploring the effects of changes in the correlation between losses, the number of consumers, and the class of utility function. We then move on in Section 3.2 to analyze allocations in the context of simulated loss data from a catastrophe reinsurer.

The main objective of this section is to compare the allocations obtained from the egalitarian equivalent approach with those derived from gradient methods that serve as the foundation for typical actuarial practice, focusing in particular on their stability properties. Specifically, we compare egalitarian equivalent allocations to those obtained from applying the gradient method to Tail Value-at-Risk (TVaR): We choose TVaR as a representative tail-risk measure, and similar stability problems are known to characterize other tail-risk measures such as Value-at-Risk and Expected Shortfall (see, e.g., Kou, Peng, and Heyde (2013)). We also include allocations derived from the Lindahl approach.

3.1. Binary Loss Distributions

Cost Allocation with Independent Losses

We now consider a simple example of an insurer with two representative consumers: consumer 1 and consumer 2, both with exponential (CARA) utility with coefficients of absolute risk aversion equal to unity, have initial wealth of 4.0 and 5.0, respectively. Both face binary loss distributions of total loss of their initial wealth (i.e., \(w^1 = L_1 = 4.0\) and \(w^2 = L_2 = 5.0\)) with the probability of 10\%. Finally, suppose that the losses are independently distributed so that the maximum aggregate loss is 9.0 with the probability of 1\%. As defined earlier, claim payments are calculated by pro rata shares in the case that the aggregate loss amount exceeds assets, and we assume that promised indemnification is set to the full amount of the loss:

\[
R^i = \min \left[ L^i, \frac{a}{\sum_{j=1}^{N} L^j} L^i \right].
\]

We can then express the expected utility functions for consumer \(i (i=1, 2)\) as:

\[
V^i = -0.1 \times 0.9 \times \exp\left\{ -\left[w^i - L^i + \min\left(L^i, a\right) - p^i\right]\right\}
- 0.1 \times 0.1 \times \exp\left\{ -\left[w^i - L^i + \min\left(L^i, \left(\frac{a}{a+5}\right) L^i\right) - p^i\right]\right\}
- 0.9 \times \exp\left\{ -(w^i - p^i)\right\},
\]

where \(p^i\) is the premium paid by consumer \(i\) and \(a\) is the asset level of the company.
We assume that the cost of holding assets is a linear function of the assets held, as in:

\[ c(a) = \tau a \quad \text{for } a > 0, \quad 0 \text{ otherwise} \]

where \( \tau \) represents the cost per asset, so that premiums follow:

\[ p_i = E[R_i^1] + z^i, \]

with \( z^1 + z^2 = \tau a \).

We can then specify the Pareto problem as:

\[ \max_{\tau, z^1, z^2} \{ V^1 + \lambda V^2 \} \]

subject to

\[ z^1 + z^2 = \tau a. \]

with \( \lambda \) being the relative weight on the second consumer.

The Pareto-optimal level of assets in this case is independent of the Pareto weight. The optimal level of assets approaches the maximum aggregate loss of 9.0 as \( \tau \) approaches zero, and it monotonically decreases as \( \tau \) increases as shown Figure 1.

The graph of the optimal asset level has two discontinuities around asset levels corresponding to the potential losses of each consumer. For \( \tau < 0.060 \), the optimal level of assets is greater than 5.0, which is sufficient to pay claims in full unless both losses happen simultaneously; at \( \tau = 0.060 \), the optimal asset level drops below 5.0, and is now insufficient to pay the loss of the second consumer. A second discontinuity happens at \( \tau = 0.191 \), where the optimal level of assets drops below 4.0 and thus becomes insufficient to pay for any single loss in full. These points of discontinuity around critical asset levels are important, because it is at these points where cost allocations under traditional methods also become discontinuous, resulting in violations of cost monotonicity.

Figure 1 also shows that the egalitarian equivalent (EE) level of asset is monotonically decreasing. As described previously, the EE level of assets is the level of assets such that, if the consumers were provided with that level without having to pay any frictional costs, they would both have the same utility as they would at the Pareto optimal asset level while paying a cost allocation. That is, the allocation rule is determined by their willingness to pay for the increase of the level of asset from the EE level to the optimal level.

Figure 2 illustrates the asset cost allocation to the first consumer with \( w^1 = L^1 = 4.0 \). Economically, any cost allocation can be justified in a Pareto sense by a particular weighting scheme (i.e., a particular level of \( \lambda \)). Our interest, however, is in stability, and we compare the proposed EE allocations (labelled EE 1 in the figure) with those produced by two other allocation methods: an allocation based on the gradient of TVaR (TVaR 1), and a Lindahl (Lindahl 1) allocation. TVaR is computed with a threshold corresponding to the point of default:
The gradient method allocates according to:

\[ z^i = \left( \frac{E[L^i | L^1 + L^2 > a]}{E[L^1 + L^2 | L^1 + L^2 > a]} \right) \times \tau \times a. \]

A Lindahl solution (see Kaneko, 1977), on the other hand, would feature cost shares based on the value placed by each consumer on the marginal unit of company assets. In other words,

\[ z^i = - \left( \frac{\partial V^i}{\partial a} \right) \times a, \]

where \( \frac{\partial V^i}{\partial a} \) reflects both the influence of assets on coverage, \( \frac{\partial V^i}{\partial z^i} \), and influence of assets on the actuarial cost component of the premium, \( \frac{\partial E[R^i]}{\partial a} \), as in:

\[ \frac{dV^i}{da} = \frac{\partial V^i}{\partial a} + \frac{\partial V^i}{\partial z^i} \frac{\partial E[R^i]}{\partial a}. \]

The problem of instability can be observed in Figure 2 for allocations based on the gradient of TVaR (labelled TVaR 1) and allocations based on the Lindahl method (Lindahl 1) around the critical asset levels, 4.0 and 5.0. For instance, a small improvement of cost function by changing \( \tau \) from 19.1% to 19.0% yields an increase in the optimal asset level from 3.98 to 4.02 and a discontinuous change in implied allocations. Under the TVaR allocation method, the first consumer’s cost allocation drops from 44.4% to 7.4%. When the optimal asset level falls between 4.0 and 5.0, the single loss of the first consumer is always paid in full and is thus no longer connected to company default. Since the first consumer’s loss matters only when two losses occur, the loss of the second consumer becomes the main contribution to TVaR. Similarly, when assets exceed 5.0, the second consumer now is fully covered in a single-loss scenario. As a result, default occurs only when both consumers experience losses, and both consumers similarly benefit from further increases in the asset level up to 9.0, so the cost allocation to the first consumer reverts to 44.4%.

Due to the instability of the allocations, the improvement in the cost structure will not necessarily be welcomed by both consumers under the TVaR gradient method. As another example, a decrease in \( \tau \) from 6.0% to 5.9% is associated with an increase in assets from 4.96 to 5.10, but the first consumer’s utility drops from -0.0288 to -0.0321 due to the spike in allocated cost, while the second consumer’s utility increases from -0.0155 to -0.0138. This illustrates that TVaR gradient allocation method fails cost monotonicity: Even though the transition involves a cost improvement, the first consumer ends up being worse off because of the reallocation of frictional cost.

Allocations based on the Lindahl mechanism also fail cost monotonicity. The shifts are less dramatic, but there are still discontinuous allocation changes around the critical asset levels. Specifically, a decrease in \( \tau \) from 19.1% to 19.0% pushes the optimal asset level beyond 4.0, but the cost allocation to the first
consumer jumps up. The reason is somewhat counterintuitive. When the asset level is below 4.0, a marginal change in the asset level has two effects. First, it improves utility by through better insurance coverage, by giving consumer 1 a larger payoff in the solo loss scenario (where she loses but consumer 2 does not); second, it reduces utility by raising the actuarial cost that must be paid in all states of the world, including the worst-case scenario when both consumers lose. The second effect trumps the former when the asset level is close to 4.0, since the consumer is already close to full coverage in the solo loss scenario. Thus, a cost improvement achieved by changing $\tau$ from 19.1% to 19.0% (which pushes the optimal asset level from just below 4.0 to just above it) is associated with a decline in utility from -0.0324 to -0.0330 for Consumer 1. For a similar reason, the cost allocation to consumer 2 jumps up when the cost improves around the critical asset level of 5.0, making Consumer 2 opposed to a drop in $\tau$ from 6.0% to 5.9%.

In contrast with the TVaR and Lindahl cost allocation rules, the EE method yields stable allocations, as evidenced by the smooth graph in Figure 2. The allocation to the first consumer declines as the cost function improves and starts increasing only when the asset exceeds 5.0. The EE mechanism guarantees that both consumers welcome any improvements to the cost structure.

Cost Allocation with Correlated Losses

Here we relax the assumption of independent losses by incorporating correlation of loss occurrence. This change affects both the optimal asset level and the cost allocation. Negatively correlated loss occurrence leads to a smaller Pareto optimal asset level and larger swings in TVaR cost allocations around the critical asset levels, due to a smaller probability of simultaneous losses (Figure 3). On the other hand, the smaller probability of simultaneous losses leads to smoother Lindahl allocations around the critical points, since the utility impact of the increase in actuarial costs (as asset levels rise in the neighbourhood of the critical points) is less pronounced when the worst-case scenario is less likely. Positively correlated loss occurrence is associated with a larger optimal asset size and smaller fluctuations in TVaR cost allocations because simultaneous losses are now more likely to happen (Figure 4). In the extreme scenario of perfect positive correlation, the TVaR allocation becomes constant. The Lindahl allocation rule also becomes stable and approaches the EE allocation for losses with perfect positive correlation.

Cost Allocation with More Consumers

Increasing the number of consumers tends to reduce the instability of cost allocations, as the proliferation of loss scenarios tends to smooth out the jumps around critical asset levels. For instance, assuming independent 100 consumers in total with 50 consumers for each consumer type, we can summarize the total share of costs for the first consumers as shown in Figure 5. The TVaR allocation rule becomes more stable, yet violations of cost monotonicity can still be detected. For instance, when the asset cost improves from 8.8% to 8.7%, the total cost share of consumer 1 increases by 1.4%, and utility falls. In contrast, both the EE allocation rule and the Lindahl allocation rules yield smooth allocations when expressed as functions of the asset cost parameter $\tau$.  

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CRRA Utility Specification

In theory, the cost monotonicity of the EE cost allocation mechanism should not depend on the choice of utility function. Here we reproduce the cost allocation rule based on CRRA utility function, specifically the log utility function, $\log(w)$. The allocation rule for the CRRA utility shown in Figure 6 appears quite different from the one obtained in the CARA utility specification (Figure 2), but both serve to contrast the jumps evident in TVaR and Lindahl allocations with the smooth transitions of the EE allocations.

3.2. Catastrophe Insurance Claims

We now apply the allocation methods to catastrophe insurance claims data. We obtained access to 50,000 simulated loss realizations from a global catastrophe reinsurer. Each of the 50,000 trials contains joint realizations for losses spanning 24 different sublines and geographic regions. To simplify calculations, we pull out a 5,000 observation subsample and aggregate the data into four lines: Earthquake, Wind, Crop/Brushfire, and Terror/Casualty. The summary statistics of our sample data is reported in Table 2. As observed, wind risk tends to have large claims and has a long right tail. Earthquake losses also have a long right tail relative to other two types with the maximum loss of 771 million. Thus, the loss distributions of risks are heterogeneous.

To compute the capital allocations, we make several assumptions. First, we assume a representative agent for each line so that the Pareto optimal asset level and the capital allocations are identified by the four representative agents’ expected utilities for 5,000 loss realizations. To be consistent with previous numerical examples, each agent has exponential utility with a risk aversion coefficient of one. Initial wealth for each agent is set to be the same as the maximum possible loss realization for each agent. As before, premiums are calculated by the expected recovery of loss plus some portion of the frictional costs of holding the assets.

The Pareto optimal level of assets is 938 million at $\tau = 0.1\%$, which is approximately the 99.3 percentile of the aggregate loss distribution but not close to the maximum aggregate loss of 1,588 million. Again, under the CARA utility function, the Pareto optimal asset level is independent of the choice of Pareto weights, and any particular cost allocation could be justified by changing the Pareto weights. The asset cost allocations under EE allocation and TVaR, for various levels of $\tau$, are summarized in Table 3. The TVaR method allocates the lion’s share of cost responsibility to the wind risk, which evidently has the largest contribution to the tail outcomes associated with default scenarios. Interestingly, Table 3 and Figure 7 reveal that the EE allocation method tends to penalize the wind risk even more heavily, which reflects the greater utility benefit the wind risk obtains from greater asset coverage. Both methods generally allocate less cost responsibility to the wind risk as $\tau$ increases: as the optimal asset level declines, the other risks become relatively more important contributors to losses in default scenarios.

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8 This data is originally used in Bauer and Zanjani (2013b), where sample statistics for the full sample and each of the sublines can be found. The four-line aggregation used here is also taken from Bauer and Zanjani (2013b).

9 By changing the risk aversion coefficient from 1 to 1.5, the Pareto optimal level of asset at $\tau = 0.1\%$ increases to 1,004 million, which is the 99.5 percentile of the aggregate loss distribution. Similarly, with log utility specification, the optimal level of asset increases to 1,073 million, which represents the 99.7 percentile of the distribution.
The jagged graph of the TVaR-based allocation to the wind risk in Figure 7 suggests violations of cost monotonicity, and these are easily verified. For instance, when $\tau$ improves from 0.15% to 0.1% (Figure 7), the cost share for wind risk increases by 4%, which translates into a utility loss for the wind line. In contrast, the smooth graph of the EE allocation testifies to the adherence of the EE mechanism to cost monotonicity. This underscores how traditional methods can yield unstable results in numerical settings, even where the number of data points is fairly large.

4. Conclusion

This paper explored egalitarian equivalence as a capital allocation concept, and argued that it is suitable for situations where the level of capital is variable but the risk portfolio is fixed. In such circumstances, the capital cost allocation problem is isomorphic with the much-studied economic problem of how to share the cost of a public good. The egalitarian equivalent allocation approach has the significant advantage of cost monotonicity, which delivers stability.

However, it must be stressed that this advantage is context-dependent: Egalitarian equivalent allocation methods are not appropriate for pricing applications where the risk portfolio is not fixed. When the problem is one of portfolio optimization, marginal cost pricing dictates the use of allocation methods, such as the Euler method in the case of risk-measure constrained portfolio optimization, even if the method produces unstable allocations. Indeed, the Euler method is likely to yield unstable allocations unless one is willing to select the risk measure specifically for stability properties.

Moulin showed that egalitarian equivalent mechanisms are the only ones that can be guaranteed to be cost monotonic in all situations, but it is possible that other methods might be admissible if further restrictions are added to nature of the cost functions. Additional restrictions might be worth exploring because the egalitarian equivalent mechanism may not be intuitive for everyone. Moulin’s terminology was evidently intended to parallel egalitarian equivalence for private good allocations (Pazner and Schmeidler, 1978). There are similarities in process: Egalitarian equivalent cost shares are found by calculating amounts that yield a particular utility distribution, while egalitarian equivalent private good allocations are found by identifying Pareto optimal allocations that match a particular utility distribution. However, the relevance of the reference point in the private case (the utility distribution associated with an economy in which all goods are shared equally) is easily and intuitively grasped, while the relevance of the corresponding reference point for the public case is less obvious. Future research may uncover other cost monotonic mechanisms in the context of more restricted settings.
References


Figure 1 Optimal Level of Assets and EE Level of Assets

Figure 2 Optimal Level of Assets and Asset Cost Allocation to Consumer 1
Figure 3 Allocation Rule for Negatively Correlated (-0.1) Losses

Figure 4 Allocation Rule for Positively Correlated (0.2) Losses
Figure 5 Allocation Rule for 100 Consumers

Figure 6 Cost Allocation Rule for 2 Consumers with Log Utility
Figure 7 EE and TVaR Cost Allocation Rule for 5000 Cat Sample
Table 1 Allocation of Asset Cost to Consumer 1

<table>
<thead>
<tr>
<th>tau</th>
<th>Optimal Asset</th>
<th>EE Asset</th>
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<td>4.958</td>
<td>3.344</td>
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<td>4.778</td>
<td>3.109</td>
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<td>2.917</td>
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Table 2 Summary Statistics of Insurance Claims (USD; Millions)

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<th>Coverage</th>
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<th>Median</th>
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<th>95th pctl</th>
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<th>Max</th>
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<td>53.93</td>
<td>9.79</td>
<td>11.52</td>
<td>29.79</td>
<td>78.67</td>
<td>306.99</td>
<td>771.29</td>
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<tr>
<td>Wind</td>
<td>138.31</td>
<td>147.95</td>
<td>81.58</td>
<td>139.52</td>
<td>295.68</td>
<td>457.00</td>
<td>786.67</td>
<td>1377.16</td>
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<tr>
<td>Crop/Brushfire</td>
<td>18.72</td>
<td>28.71</td>
<td>10.18</td>
<td>18.06</td>
<td>28.28</td>
<td>62.38</td>
<td>166.33</td>
<td>347.54</td>
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<tr>
<td>Terror/Casualty</td>
<td>10.75</td>
<td>15.77</td>
<td>5.76</td>
<td>8.69</td>
<td>19.90</td>
<td>40.21</td>
<td>100.96</td>
<td>158.63</td>
</tr>
<tr>
<td>Total</td>
<td>190.62</td>
<td>164.47</td>
<td>127.23</td>
<td>210.22</td>
<td>387.31</td>
<td>543.32</td>
<td>892.13</td>
<td>1588.23</td>
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Table 3 Asset Cost Allocation for Catastrophe Insurance Claims

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<th>EE Share</th>
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<td></td>
<td>EQ</td>
<td>Wind</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Terror/Casualty</td>
<td>EQ</td>
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<tr>
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<td>132.9</td>
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<td>0.880</td>
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<td>311.3</td>
<td>87.3</td>
<td>0.106</td>
<td>0.860</td>
</tr>
<tr>
<td>0.030</td>
<td>262.8</td>
<td>22.1</td>
<td>0.137</td>
<td>0.812</td>
</tr>
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APPENDIX

Proof of Theorem 1 (Moulin, 1987)

Prove finiteness of $a^*$:

Risk aversion and non-trivial loss distributions implies that

$$V^i(a, 0) > V^i(0, 0) \quad \forall \ i \in \{1, \ldots, N\}, a > 0$$

We take an increasing sequence $\hat{a}_t$ such that

$$\lim_{t \to \infty} \hat{a}_t = a^* \quad (A.1)$$

By way of contradiction, suppose

$$\lim_{t \to \infty} \hat{a}_t = a^* = \infty \quad (A.2)$$

By definition of $a^*$, we know there is an associated sequence of feasible allocations $(a_t, z^1_t, \ldots, z^N_t)$ satisfying:

$$V^i(\hat{a}_t, 0) \leq V^i(a_t, z^i_t) \quad \forall \ i \in \{1, \ldots, N\}. \quad (A.3)$$

Suppose this sequence $a_t$ is also unbounded. Then we can find a subsequence, denoted $\hat{a}_t$, that converges to infinity.

Denote the upper bounds of the loss distributions by $\bar{L}_i$. An intermediate value argument, which we can apply due to the observation that $V^i(a_t, 0) \geq V^i(0, 0) \geq V^i(a_t, \bar{L}_i) \quad \forall \ i, \ t$, and the continuity of the utility functions, imply that for each $i$ there exists a function $z_i(.)$ satisfying:

$$V^i(\hat{a}_t, \hat{z}^i_t) \geq V^i(\hat{a}_t, 0) \geq V^i(0, 0) = V^i(\hat{a}_t, z_i(\hat{a}_t)) \quad \forall \ i, \ t \quad (A.4)$$

Because the loss distributions are bounded, assets have no value beyond a certain point, so $z_i(.)$ is bounded from above, so (A.4) implies that $\hat{z}^i_t$ must be bounded from above as well.

It follows that

$$\lim_{t \to \infty} \sup_{i \in \{1, \ldots, N\}} \left( \frac{\hat{z}_t^i}{\hat{a}_t} \right) \leq 0 \quad \forall \ i \in \{1, \ldots, N\},$$

and moreover that

$$\lim_{t \to \infty} \sup_{i \in \{1, \ldots, N\}} \left( \frac{\sum \hat{z}_t^i}{\hat{a}_t} \right) \leq 0$$
However, notice that feasibility and the convexity of the cost function implies that:

\[
\limsup_{t \to \infty} \left\{ \frac{\sum z^i_t}{\hat{a}_t} \right\} = \limsup_{t \to \infty} \left\{ \frac{c(\hat{a}_t)}{\hat{a}_t} \right\} > 0
\]

which contradicts the previous result. Thus the sequence \( \hat{a}_t \) must be bounded, meaning that

\[
\lim_{t \to \infty} \hat{a}_t = q < \infty
\]

By assumption of unboundedness, \( \hat{a}_t > q \) for large enough \( t \), so (A.3) then implies that \( z^i_t \leq 0 \) for all \( i \in \{1, ..., N\} \), which violates feasibility, a contradiction indicating that \( a^* < \infty \).

Prove \( V^i(a^*, 0) = V^i(\tilde{a}, \tilde{z}^i) \ \forall \ i \in \{1, ..., N\} \):

First, we prove that a feasible allocation \( (a, z^1, ..., z^N) \) satisfies

\[
V^i(a^*, 0) \leq V^i(a, z^i) \ \forall \ i \in \{1, ..., N\}
\]  \hspace{1cm} (A.5)

We take a bounded increasing sequence \( \hat{a}_t \) such that

\[
\lim_{t \to \infty} \hat{a}_t = a^* < \infty
\]

associated sequence of feasible allocations \( (a_t, z^1_t, ..., z^N_t) \) satisfying (A.3). We know from the previous step that all elements of this sequence are bounded, so the associated sequence must have a convergent subsequence. Define \( \tilde{\Omega} \subset \Omega \) as:

\[
\tilde{\Omega} = \left\{ (a, z^1, ..., z^N) \mid Q \times \sum_i L_i \geq a \geq 0, \sum_i z^i = c(a), z^i \in [-Q \times \max(L^1, ..., L^N), Q \times \max(L^1, ..., L^N)] \ \forall \ i \in \{1, ..., N\} \right\}
\]

where \( Q \) is any number greater than 1. Notice that \( (a_t, z^1_t, ..., z^N_t) \in \tilde{\Omega} \ \forall t \), since \( \hat{a}_t \geq 0 \) and any feasible allocation lying outside \( \tilde{\Omega} \) would involve a violation of (A.3). Moreover, since \( \tilde{\Omega} \) is closed, any convergent subsequence of \( (a_t, z^1_t, ..., z^N_t) \) converges to a limit point of \( \tilde{\Omega} \), so (A.5) is satisfied.

Given \( a^* \geq 0 \), note that egalitarian equivalence implies that \( \tilde{z}^i \not\in \{-Q \times \max(L^1, ..., L^N), Q \times \max(L^1, ..., L^N)\} \) for all \( i \in \{1, ..., N\} \). To see why, consider the case where \( \tilde{z}^i = Q \times \max(L^1, ..., L^N) \) for some \( i \). Then it follows that:

\[
V^i(a^*, 0) \geq V^i(0,0) \geq Eu^i(w^i - \tilde{L}^i) > V^i(\tilde{a}, \tilde{z}^i)
\]
Let strict. Then there exists some with strict equality for at least one of the feasible alternative allocation equivalent level of public good production as 

Proof: Denote an egalitarian equivalent allocation as 

This contradiction implies that \( \bar{z}^i \) will always lie in the interior of \( \{-Q \times \max\{L^1, ..., L^N\}, Q \times \max\{L^1, ..., L^N\}\} \).

Moving on, by way of contradiction, suppose that \( V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i) \) is not satisfied \( \forall i \). This implies that there exists some nonempty subset of consumers (which we will denote by \( M \)) such that:

\[
V^k(a^*, 0) < V^k(\bar{a}, \bar{z}^k) \quad \forall k \in M
\]

Note that \( M \) cannot be equivalent to \( \{1, ..., N\} \) (i.e., \( V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i) \) must hold for some \( i \)), as this would contradict the egalitarian equivalence of \( a^* \) since we could increase \( a^* \) by some amount if \( V^k(a^*, 0) < V^k(\bar{a}, \bar{z}^k) \) \( \forall k \in \{1, ..., N\} \). So we consider a complementary set \( L = \{1, ..., N\}/M \) with

\[
V^j(a^*, 0) = V^j(\bar{a}, \bar{z}^j) \quad \forall j \in L
\]

But this is also incompatible with the egalitarian equivalence of \( a^* \) since, given that all cost shares are interior to the choice set, we could form a new feasible allocation, \( (\bar{a}, \bar{z}^1, ..., \bar{z}^N) \), where \( \bar{a} = \bar{a} \) and we have subtracted some small amount from each of cost shares of all agents in \( L \) and divided the sum total of those deductions among the agents in \( M \) so that:

\[
V^k(a^*, 0) < V^k(\bar{a}, \bar{z}^k) \quad \forall k \in \{1, ..., N\}.
\]

This contradiction implies that \( V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i) \) must be satisfied for all \( i \in \{1, ..., N\} \).

Proof of Theorem 2

Proof: Denote an egalitarian equivalent allocation as \( (\bar{a}, \bar{z}^1, ..., \bar{z}^N) \) and the associated egalitarian equivalent level of public good production as \( a^* \). Suppose it is not Pareto efficient. Then there exists a feasible alternative allocation \( (\bar{a}, \bar{z}^1, ..., \bar{z}^N) \) satisfying:

\[
V^i(a^*, 0) = V^i(\bar{a}, \bar{z}^i) \leq V^i(\bar{a}, \bar{z}^i) \quad \forall i \in \{1, ..., N\},
\]

with strict equality for at least one of the \( i \)'s. Let \( k \) index one of the agents for whom the inequality is strict. Then there exists some \( \varepsilon > 0 \) such that

\[
V^k(\bar{a}, \bar{z}^k) < V^k(\bar{a}, \bar{z}^k + \varepsilon) < V^k(\bar{a}, \bar{z}^k)
\]

Let
\[
\tilde{z}^i = \tilde{z}^i - \frac{\varepsilon}{N-1} \quad \forall i \neq k
\]

\[
\tilde{z}^k = \tilde{z}^k + \varepsilon
\]

Note that 

\[
(a, \tilde{z}^1, ..., \tilde{z}^N) \in \Omega
\]

But since utility is strictly decreasing in the second argument

\[
V^i(a^*, 0) = V^i(\hat{a}, \tilde{z}^i) < V^i(\tilde{a}, \tilde{z}^i) \quad \forall i \in \{1, ..., N\},
\]

which is inconsistent with \(a^*\) being the egalitarian equivalent level of public good production.

**Proof of Theorem 3**

Suppose not. Then there exist two cost functions \(c_1(.)\) and \(c_2(.)\), with \(c_1 \leq c_2\), but where the associated egalitarian equivalent levels of assets, \(a_1^*\) and \(a_2^*\), satisfy \(a_1^* < a_2^*\). Let \((a_2, z_2^1, ..., z_2^N)\) be an egalitarian equivalent allocation assigned by the mechanism under \(c_2(.)\) and \((a_1, z_1^1, ..., z_1^N)\) an egalitarian equivalent allocation assigned under \(c_1(.)\). Note that:

\[
\left( a_2, z_2^1 - \frac{c_2(a_2) - c_1(a_2)}{N}, ..., z_2^N - \frac{c_2(a_2) - c_1(a_2)}{N} \right)
\]

is a feasible allocation under \(c_1(.)\). Egalitarian equivalence, together with \(c_1 \leq c_2\), implies that

\[
V^i(a_2^*, 0) = V^i(a_2, z_2^i) \leq V^i\left( a_2, z_2^i - \frac{c_2(a_2) - c_1(a_2)}{N} \right) \quad \forall i \in \{1, ..., N\}.
\]

But egalitarian equivalence, together with \(a_1^* < a_2^*\), implies that

\[
V^i(a_2^*, 0) > V^i(a_1^*, 0) = V^i(a_1, z_1^i) \quad \forall i \in \{1, ..., N\}.
\]

Putting these together yields

\[
V^i(a_1, z_1^i) = V^i(a_1^*, 0) < V^i\left( a_2, z_2^i - \frac{c_2(a_2) - c_1(a_2)}{N} \right) \quad \forall i \in \{1, ..., N\},
\]

which contradicts the supposition that \(a_1^*\) is egalitarian equivalent under \(c_1(.)\).