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An Extended Lee-Carter Model for Mortality Differential by Long-Term Care Status

Abstract

This paper aims to propose a new methodology to forecast mortality rates of the long-term care (LTC) population with longevity risk. A major obstacle to devising such a method is lack of data on the number of deaths in LTC populations, which prevents us from using the conventional mortality model such as the Lee-Carter model. To overcome this difficulty, we propose an extended Lee-Carter model with a term representing mortality differential due to LTC status which does not require the data on the number of deaths in LTC populations. We apply the proposed model to the data from the Japanese long-term care insurance system. Our preliminary results shows that the proposed method captures the heterogeneity in the mortality rate between the LTC statuses properly. We plan to deliver a more detailed investigation at the conference.

1 Introduction

Increased human lifetime is accompanied by a greater chance of becoming disabled, which could require significant additional costs for long-term care (LTC henceforth). Therefore, needs for LTC have been increased and financing its cost has been an important topic for developed economies. Impaired annuities, enhanced annuities, and life annuities that pay out a higher rate once an insured becomes LTC disabled may be considered the solutions, which share the characteristic that pays more to insured with lower life expectancy which is evaluated based on insured's particular health problems. Thus, the key element of the solutions is how the mortality is related to health states. However, study on the complicated mortality dynamics is limited due to lack of data.

Using Italian national data on health and mortality, Levantesi and Menzietti (2012) show that the premium rates variability is higher for LTC insurance than for enhanced annuity with increased annuity rate for LTC disabled. This demonstrates natural hedging between life annuities and LTC insurance. Gourieroux and Lu (2013) propose an approach to model the entry into LTC as a unobserved jump in the mortality intensity and to derive LTC hazard and mortality rates from the underlying mortality data with or without reference to LTC data.

This paper aims to propose a new method to forecast mortality rates of the long-term care (LTC) population with longevity risk. A major obstacle to devising such a method is lack of data on the number of deaths in LTC population, which prevents us from using the conventional mortality model such as the Lee-Carter model. To overcome this difficulty, we propose an extended Lee-Carter model with a term representing mortality differential due to LTC status which does not require the data on the number of deaths in LTC populations. We apply the proposed model to the data from the Japanese long-term care insurance system. Our preliminary results shows that the proposed method captures the heterogeneity in the mortality rate between the LTC statuses properly.

The paper is organized as follows. In Section 2, we model mortality differentials for LTC populations in the Lee-Carter model. The estimation approach is proposed in Section 3. In Section 4 we implement the model for prediction purpose using the Japanese mortality and LTC data. Section 5 concludes.

2 Lee-Carter Modeling for differential by LTC status

Let E_{xtj} and D_{xtj} denote the population size and the number of deaths for age $x(x = x_{\min}, \dots, x_{\max})$ with LTC status j in year $t(t = t_{\min}, \dots, t_{\max})$. Here j ranges from 1(=least severe status) up to J(=most severe status). Then the total population size and the total number of deaths for age x in year t is given by

$$D_{xt} = \sum_{j=1}^{J} D_{xtj}, E_{xt} = \sum_{j=1}^{J} E_{xtj}.$$

D_{xt}, $E_{xtj}(j = 1, \dots, J)$ are observed, but D_{xtj} are not available.

Suppose that the force of mortality of an individual aged x at time t in LTC status j follows the Lee-Carter law

$$\mu_{xtj} = \exp\{\alpha_{xj} + \beta_{xj}\kappa_{tj}\} \tag{1}$$

Here we assume that α_{xj}, which represents the age effect of LTC status j, is written as

$$\alpha_{xj} = \gamma_x + \eta_j$$

We expect that the mortality rate becomes higher as the LTC status gets worse on average. Thus we impose a monotonicity constraint:

$$\eta_1 \leq \eta_2 \leq \dots \leq \eta_J, \tag{2}$$

which makes sure that α_{xj} is monotone increasing with j for each x.

κ_{tj} may be interpreted as the common period effect across age due to factors such as medical skills or public health. Thus we assume that it does not depend on j. β_{xj} represents the sensitivity of the mortality to changes in κ_t. β_{xj} may depend on both x and j in general. Here for simplicity we assume that it depends only on x. Thus (1) becomes

$$\mu_{xtj} = \exp\left\{\gamma_x + \eta_j + \beta_x\kappa_t\right\}, \tag{3}$$

Then we assume that the mortality of the total population of age x in year t is given as the geometric average of (3):

$$\mu_{xt} = \prod_{j=1}^{J} (\mu_{xtj})^{w_{xtj}} = \exp\left\{\gamma_x + \sum_j w_{xtj}\eta_j + \beta_x\kappa_t\right\},$$

where $w_{xtj} \equiv E_{xtj}/E_{xt}$. If all the LTC status parameters η_j are equal:

$$\eta_1 = \eta_2 = \dots = \eta_J = \eta,$$

then we have

$$\mu_{xt} = \exp\left\{\gamma_x + \eta + \beta_x \kappa_t\right\},$$

which is the standard Lee-Carter model by writing α_x for $\gamma_x + \eta$. Thus the proposed model is an extension of the standard Lee-Carter model so as to incorporate mortality differential by LTC status. It is interesting to note that the proposed model corresponds to the common factor model of Li and Lee (2005) and the stratified Lee-Carter model in Butt and Haberman (2009). Our model differs with these models in two respects. One is that our model does not require the death number data D_{xtj}. The other is that we impose the monotonicity condition (2).

3 Estimation

3.1 Normal model

Let

$$m_{xt} \equiv \frac{D_{xt}}{E_{xt}}$$

denote the mortality rate. To estimate $\{\gamma_x, \eta_j, \beta_x, \kappa_t\}$, we assume that the logarithm of the mortality $y_{xt} \equiv \log m_{xt}$ follows a normal model

$$\begin{aligned} y_{xt} &= \log \mu_{xt} + \varepsilon_{xt}, \quad \varepsilon_{xt} \overset{i.i.d.}{\sim} N(0, \sigma^2) \\ &= \gamma_x + \sum_{j=1}^{J} w_{xtj}\eta_j + \beta_x \kappa_t + \varepsilon_{xt} \\ &= \gamma_x + \boldsymbol{w}'_{xt}\boldsymbol{\eta} + \beta_x \kappa_t + \varepsilon_{xt} \end{aligned} \tag{4}$$

where ε_{xt}'s are independent errors assumed to be identically distributed as the normal distribution with mean 0 and the variance σ^2 and

$$\boldsymbol{w}_{xt} \equiv \begin{pmatrix} w_{xt1} \\ w_{xt2} \\ \vdots \\ w_{xtJ} \end{pmatrix}, \boldsymbol{\eta}_x \equiv \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_J \end{pmatrix}.$$

In a matrix form (4) is written as

$$\boldsymbol{y}_x = \gamma_x \mathbf{1}_{N_T} + \boldsymbol{W}_x \boldsymbol{\eta} + \beta_x \boldsymbol{\kappa} + \boldsymbol{\varepsilon}_x, \tag{5}$$

where

$$\boldsymbol{y}_x = \begin{pmatrix} y_{xt_{\min}} \\ \vdots \\ y_{xt_{\max}} \end{pmatrix}, \mathbf{1}_{N_T} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \boldsymbol{W}_x = \begin{pmatrix} \boldsymbol{w}'_{xt_{\min}} \\ \vdots \\ \boldsymbol{w}'_{xt_{\max}} \end{pmatrix}, \boldsymbol{\kappa} = \begin{pmatrix} \kappa_{t_{\min}} \\ \vdots \\ \kappa_{t_{\max}} \end{pmatrix}, \boldsymbol{\varepsilon}_x = \begin{pmatrix} \varepsilon_{xt_{\min}} \\ \vdots \\ \varepsilon_{xt_{\max}} \end{pmatrix}.$$

and $N_T \equiv t_{\max - t_{\min} + 1}$.

3.2 Maximum likelihood under the equality and inequality constraints

As in the standard Lee-Carter model we make the assumptions

$$\sum_x \beta_x = 1, \ \sum_t \kappa_t = 0 \tag{6}$$

for the parameter identification. Then the maximum likelihood estimation is equivalent to minimizing

$$\begin{aligned} f(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}) &= \sum_x \sum_t \left(y_{xt} - \gamma_x - \boldsymbol{w}'_{xt}\boldsymbol{\eta} - \beta_x \kappa_t\right)^2 \\ &= \sum_x (\boldsymbol{y}_x - \gamma_x \mathbf{1}_{N_T} - \boldsymbol{W}_x\boldsymbol{\eta} - \beta_x\boldsymbol{\kappa})'(\boldsymbol{y}_x - \gamma_x \mathbf{1}_{N_T} - \boldsymbol{W}_x\boldsymbol{\eta} - \beta_x\boldsymbol{\kappa}) \quad (7) \end{aligned}$$

subject to the constraints (2) and (6).

To solve this constrained minimization problem we consider a Lagrangian function

$$L(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}; \lambda_1, \lambda_2) = \frac{1}{2} f(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}) + \lambda_1 \left(\sum_x \beta_x - 1\right) + \lambda_2 \left(\mathbf{1}'_{N_T}\boldsymbol{\kappa}\right) + \sum_{j=1}^{J-1} \mu_j(\eta_j - \eta_{j+1}),$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2, \ldots, \mu_{J-1}$ are Lagrangian multiples. Then the Karush-Kuhn-Tucker (KTT) conditions are:

$$\begin{aligned} &\frac{\partial L}{\partial \gamma_x} = 0 \\ &\frac{\partial L}{\partial \boldsymbol{\eta}} = 0 \\ &\frac{\partial L}{\partial \beta_x} = 0 \\ &\sum_x \beta_x = 1 \\ &\sum_t \kappa_t = 0 \\ &\mu_1 \geq 0,\ \mu_2 \geq 0,\ \ldots,\ \mu_{J-1} \geq 0 \\ &\mu_1(\eta_1 - \eta_2) = 0, \mu_2(\eta_2 - \eta_3) = 0, \ldots, \mu_{j-1}(\eta_{J-1} - \eta_J) = 0, \end{aligned}$$

When J is small, we can solve the constrained minimization problem directly from the KTT conditions. For example, suppose $J = 4$ and let $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$ denote the optimal values of $\eta_1, \eta_2, \eta_3, \eta_4$. Then we need to consider the following eight cases:

Case 1 $\eta_1^* < \eta_2^* < \eta_3^* < \eta_4^*$

Then it holds that $\mu_1 = \mu_2 = \mu_3 = 0$ and the minimization problem will be without the monotonicity condition.

Case 2 $\eta_1^* < \eta_2^* < \eta_3^* = \eta_4^*$

Then it holds that $\mu_1 = \mu_2 = 0$ and the inequality constraint $\eta_3 \leq \eta_4$ becomes the equality constraint $\eta_3 = \eta_4$. Thus we merge the LTC statuses 3 and and 4 into one status, $\tilde{3} = \{3, 4\}$, say, and solve the minimization problem for the three LTC statuses 1, 2 and $\tilde{3}$ without the monotonicity restraint.

Case 3 $\eta_1^* < \eta_2^* = \eta_3^* < \eta_4^*$

Then it holds that $\mu_1 = \mu_3 = 0$ and the inequality constraint $\eta_2 \leq \eta_3$ becomes the equality constraint $\eta_2 = \eta_3$ Thus we merge LTC statuses 2 and and 3 into one status, $\tilde{2} = \{2, 3\}$, say, and solve the minimization problem for the three LTC statuses 1, $\tilde{2}$ and 3 without the monotonicity restraint.

Case 4 $\eta_1^* = \eta_2^* < \eta_3^* < \eta_4^*$

Then it holds that $\mu_2 = \mu_3 = 0$ and the inequality constraint $\eta_1 \le \eta_2$ becomes the equality constraint $\eta_1 = \eta_2$. Thus we merge LTC statuses 1 and and 2 into one status, $\tilde{1} = \{1, 2\}$, say, and solve the minimization problem for the three LTC statuses $\tilde{1}$, 2 and 3 without the monotonicity restraint.

Case 5 $\eta_1^* < \eta_2^* = \eta_3^* = \eta_4^*$

Then it holds that $\mu_1 = 0$ and the inequality constraint $\eta_2 \le \eta_3 \le \eta_4$ becomes the equality constraint $\eta_2 = \eta_3 = \eta_4$.Thus we merge LTC statuses 2, 3 and 4 into one status, $\tilde{2} = \{2, 3, 4\}$, say, and solve the minimization problem for the three LTC statuses and 1 and $\tilde{2}$ without the monotonicity restraint.

Case 6 $\eta_1^* = \eta_2^* < \eta_3^* = \eta_4^*$

Then it holds that $\mu_2 = 0$ and the inequality constraints $\eta_1 \le \eta_2$ and $\eta_3 \le \eta_4$ become the equality constraints $\eta_1 = \eta_2$ and $\eta_3 = \eta_4$. Thus we merge LTC statuses 1 and 2 into one status $\tilde{1} = \{1, 2\}$, say, and LTC statuses 3 and 4 into one status $\tilde{3} = \{3, 4\}$, say, and solve the minimization problem for the three LTC statuses and $\tilde{1}$, 2 and $\tilde{3}$ without the monotonicity restraint.

Case 7 $\eta_1^* = \eta_2^* = \eta_3^* < \eta_4^*$

Then it holds that $\mu_3 = 0$ and the inequality constraint $\eta_1 \le \eta_2 \le \eta_3$ becomes $\eta_1 = \eta_2 = \eta_3$. Thus we merge LTC statuses 1, 2 and 3 into one, $\tilde{1} = \{1, 2, 3\}$, say, and and solve the minimization problem for the three LTC statuses and $\tilde{1}$, 2 and $\tilde{3}$ without the monotonicity restraint.

Case 8 $\eta_1^* = \eta_2^* = \eta_3^* = \eta_4^*$

Then it holds that $\eta_1 = \eta_2 = \eta_3 = \eta_4$. Thus we merge all the status into one and solve the minimization problem, which is equivalent to the standard Lee-Carter model.

Let f_k denote the minimized value of (7) for case k and let D denote the subclass of the eight cases for which the monotonicity condition (2) holds. Then we choose the one that minimizes f_k among the cases for which the monotonicity condition (2) satisfies.

The minimization problem for each case is solved by considering a Lagrangian function of the form

$$L(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}; \lambda_1, \lambda_2) = \frac{1}{2} f(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}) + \lambda_1 \left(\sum_x \beta_x - 1 \right) + \lambda_2 \left(\mathbf{1}'_{N_T} \boldsymbol{\kappa} \right). \quad (8)$$

The solution to this minimization is given by the following algorithm:

Algorithm for the estimation

1. Let the initial values of γ_x, β_x, $\boldsymbol{\kappa}$ be $\gamma_x^{(0)}$, $\beta_x^{(0)}$, $\boldsymbol{\kappa}^{(0)}$. We may use the estimates of the standard Lee-Carter model as the initial values.

2. For each $\nu = 1, 2, \dots$

   (a) Updating $\boldsymbol{\eta}$

$$\boldsymbol{\eta}^{(\nu)} = \left(\sum_x \boldsymbol{W}'_x \boldsymbol{W}_x\right)^{-1} \sum_x \boldsymbol{W}'_x \left(\boldsymbol{y}_x - \gamma_x^{(\nu-1)} \mathbf{1}_{N_T} - \beta_x^{(\nu-1)} \boldsymbol{\kappa}^{(\nu-1)}\right)$$

   (b) Updating γ

$$\gamma_x^{(\nu)} = \frac{1}{N_T} \sum_t y_{xt} + \frac{1}{N_T} \sum_t \boldsymbol{w}'_{xt} \boldsymbol{\eta}^{(\nu)}$$

   (c) Updating $\boldsymbol{\beta}$

$$\beta_x = \frac{\boldsymbol{\kappa}^{(\nu-1)'}(\boldsymbol{y}_x - \gamma_x^{(\nu)} - \boldsymbol{W}_x \boldsymbol{\eta}^{(\nu)})}{\boldsymbol{\kappa}^{(\nu-1)'} \boldsymbol{\kappa}^{(\nu-1)}}$$
$$\beta_x^{(\nu)} = \beta_x - \frac{1}{N_X} \sum_x \beta_x + \frac{1}{N_X}$$

   (d) Updating κ

$$\boldsymbol{\kappa} = \frac{\sum_x \beta_x (\boldsymbol{y}_x - \gamma_x^{(\nu)} - \boldsymbol{W}_x \boldsymbol{\eta}^{(\nu)})}{\sum_x (\beta_x^{(\nu)})^2}$$
$$\boldsymbol{\kappa}_x^{(\nu)} = \boldsymbol{\kappa} - \frac{1}{N_T} \sum_t \kappa_t$$

3. We repeat the above until each parameter values converges.

Another way to implement the monotonicity constraint (2) is to add the penalty term for the monotonicity to (8) as discussed in Tibshirani, Hoefling and Tibshirani (2011). That is, we consider to minimize

$$\frac{1}{2} f(\{\gamma_x\}, \boldsymbol{\eta}, \{\beta_x\}, \boldsymbol{\kappa}) + \lambda_1 \left(\sum_x \beta_x - 1\right) + \lambda_2 \left(\mathbf{1}'_{N_t} \boldsymbol{\kappa}\right) + \lambda_3 \sum_{j=1}^{J-1} (\eta_{j+1} - \eta_j)_+$$

with $x+$ indicating the positive part, $x+ \equiv x \cdot 1(x > 0)$. When letting μ_3 to ∞, we obtain the solution to to the original constrained minimization problem.

4 Application

4.1 Data

We use the data from the public long-term care insurance system in Japan which started in April 2000 to deal with the increase in demand for the nursery care associated with the rapid ageing. Under the system those in need of nursery care are categorized into seven stages: required support 1 and 2 and required care 1 to 5. For the present analysis we divide the total population into four LTC status. LTC status 1 corresponds to people not in need of nursery care. LTC status 2 consists of required support 1, 2 and required care 1. LTC status 3 consists of required care 2 and 3. LTC status 4 consists of required care 4 and 5. We regard the population size in each LTC status as the number of recipients in the same LTC status. For years from 2001 to 2014 the numbers of recipients in each stage are available for age class of 40-65, 65-to 69, 70-74, 75-79, 80-84, 85-89 and 90+. We linearly interpolate the numbers of recipients for age class to obtain the numbers of recipients at each age.

4.2 Results

The values of the objective function (7) are reported in Tables 1 and 2. The corresponding values of η_j's are listed in Tables 3 and 4. We see that the objective function is minimized for case 5 among the cases for which the monotonicity condition (2) satisfies both in the male and the female populations. Case 5 consists of two LTC statuses, one corresponding to people not in need of nursery care and the other corresponding to people in need of nursery care.

The estimated parameters for case 5 are depicted in Figures 1 and 2. We see a clear mortality differential between the two LTC statuses. It is interesting to observe that the standard Lee-Carter model and the extended model give almost identical values for β_x and κ_t.

case	f	rank
1	1.047231	1
2	1.070003	7
3	1.060944	3
4	1.049372	2
5	1.068317	5
6	1.068585	6
7	1.066173	4
8	1.072776	8

Table 1: Objective function (male)

case	f	rank
1	1.497592	1
2	1.543154	5
3	1.534138	3
4	1.520015	2
5	1.544488	7
6	1.544018	6
7	1.540497	4
8	1.550396	8

Table 2: Objective function (female)

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
η_1	-0.001409291	-0.001085932	-0.000955051	-0.001256613	-0.001389238	-0.001350959	-0.001591836
η_2	1.248034019	-2.531006585	-3.023723609	-9.845954612	0.441403616	0.651495542	1.982023312
η_3	-11.20297417	1.876465066	9.717656492	16.34062516			
η_4	16.90778509						

Table 3: Estimated η_j for each case (male)

	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
η_1	-0.002369824	-0.000789423	-0.000369216	-0.000285058	-0.001849915	-0.001678146	-0.00181043
η_2	6.064283856	-1.585601349	-1.548646565	-6.723224379	0.447572092	0.609250823	1.37522913
η_2	-16.43970076	1.105303365	3.547057579	7.308822132			
η_2	12.74692923						

Table 4: Estimated η_j for each case (female)

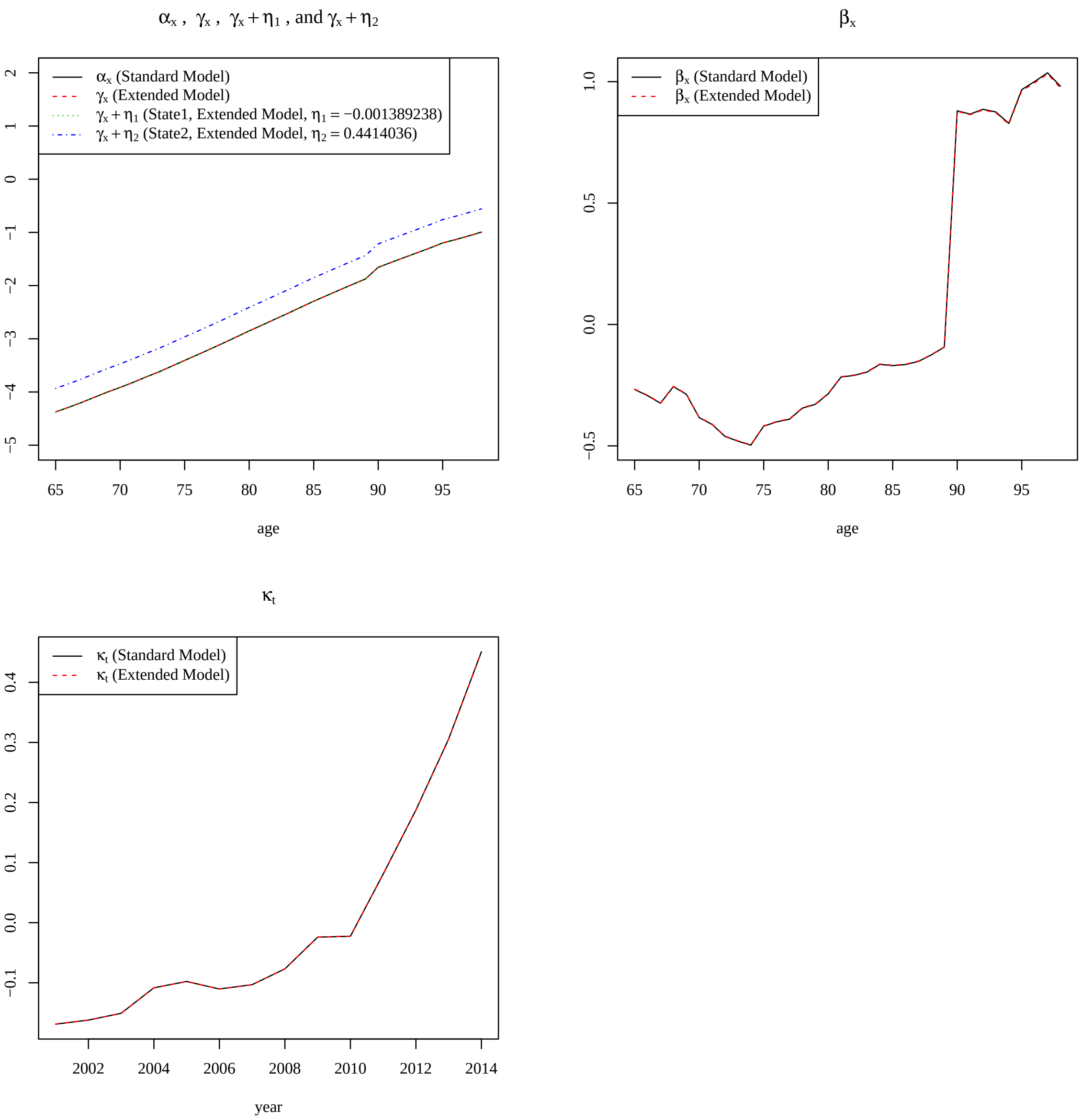

Figure 1: Case 5 (male)

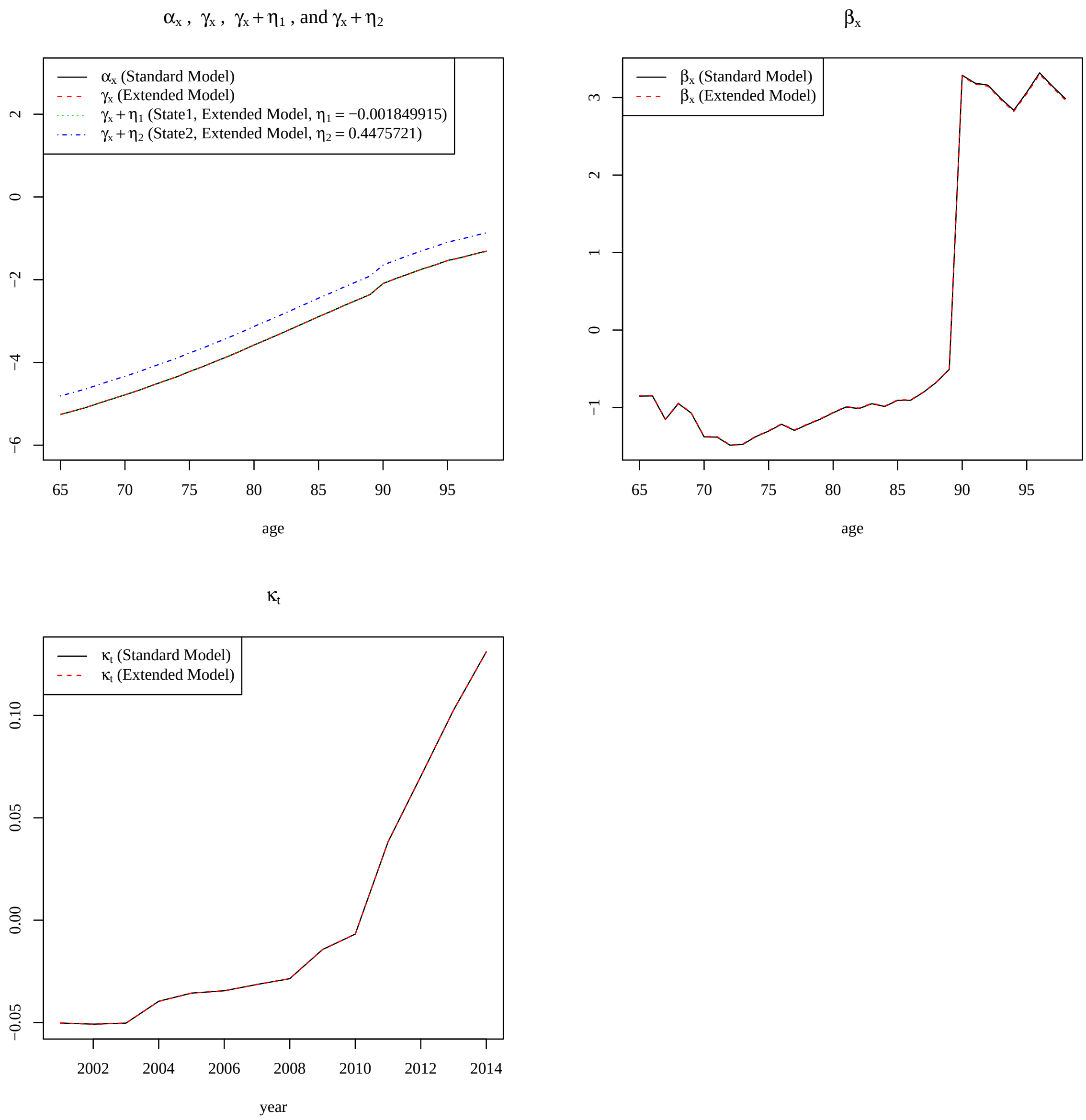

Figure 2: Case 5 (female)

4.3 Forecasting mortality rates

We plan to forecast mortality rates based on time series prediction of the calendar parameters κ_t. This can be written as follows:

$$\dot{\mu}_{x,t+k,j} = \exp\{\hat{\gamma}_x + \hat{\eta}_j + \hat{\beta}_x \dot{\kappa}_{t+k}\}, k = 1, 2, \ldots$$

where $\dot{\kappa}_{x+kj}$ represents the forecast period effects. The most common choice for time series extrapolation methods applied in the Lee-Carter framework are the ARIMA(p, d, q) processes, which is

$$\left(1 - \sum_{i=1}^{p} \phi_i L^i\right) (1 - L)^d \kappa_t = \delta + \left(1 + \sum_{i=1}^{q} \theta_i L^i\right) \varepsilon_t$$

where L represents the shift operator and $\{\varepsilon_t\}$ are error terms. In the majority of applications ARIMA$(0, 1, 0)$, the random walk with drift, is the standard choice, which can be expressed as:

$$\kappa_t = \kappa_{t-1} + \delta + \varepsilon_t$$

Estimated κ_t for the present analysis given in Figures 1 and 2 do not look like a random walk. We are currently investigating which model is suitable for our purpose.

5 Concluding remarks

In this paper we have proposed an extended Lee-Carter model with a term representing mortality differential by LTC status. We apply the proposed model to the data from the Japanese long-term care insurance system. Our preliminary results have demonstrated the heterogeneity in the mortality rate between the LTC statuses. We plan to deliver a more detailed investigation at the conference.

One problem with the present approach is that the parameter uncertainty is not taken into consideration. To deal with this problem, we may formulate the whole implementation by a Bayesian framework as discussed in Kogure and Kurachi (2010).

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