Equilibrium recoveries in insurance markets with limited liability

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Abstract

This paper studies optimal insurance in equilibrium in case the insurer is protected by limited liability. We focus on the optimal allocation of remaining assets in default. Perfect pooling of risk is optimal, but a protection fund is needed to charge levies to policyholders with low losses. If policyholders cannot be forced *ex post* to pay a levy, we show that the constrained equal loss rule is optimal. This rule gained particular interest in the literature on rationing. We show existence of an equilibrium in the market, and study the welfare losses if other recovery rules are used in default. Moreover, we show that in absence of a monitoring device, the insurer will always invest all its assets in the risky technology. We show that the welfare losses of choosing a wrong recovery rule can be substantial for the insurer.

1 Introduction

This paper studies optimal recoveries in insurance, and their effects on prices in equilibrium. We use an agency model, where a mutual insurer is protected by limited liability. In case of a default, the remaining assets of the insurer are (at least partially) allocated to the policyholders. We show that using a proportional method to allocate the recoveries yields welfare losses in the economy. Moreover, we characterize the optimal method instead. In rationing, this optimal method is called a constrained equal loss rule (see, e.g., Moulin, 2002, Thomson, 2003).

A rationing problem describes the situation in which we allocate a given amount (often referred to as estate) among a group of claimants when the available amount is not enough to satisfy all their claims. A rationing solution calculates shares for claimants such that 1) no agent gets more than its claim, and 2) all get a non-negative share. For rationing problems in practice and rationing solutions see, e.g., O'Neill (1982), Aumann and Maschler (1985), Moulin (2000), or the overviews of

Moulin (2002), and Thomson (2003). In a natural way any realization of a default situation with limited liability is related mathematically to a rationing problem with the indemnities as claims and the realized assets as the size of the estate. Then, any rationing solution can be taken to define a solution to the default situation with limited liability. Habis and Herings (2013) and Koster and Boonen (2014) study stochastic rationing problems in risk sharing problems. We apply the concept of stochastic rationing to an insurance setting with limited liability.

Initially, Doherty and Schlesinger (1990), Cummins and Mahul (2003) and Bernard and Ludkovski (2012) study insurance contract design with limited liability by modeling default as an exogenous event that is correlated with the risk of a policyholder. We follow the approach of Filipović et al. (2015) to study optimal risk taking and premia of an insurer in equilibrium. They study this problem with one insurer and one policyholder. Moreover, Biffis and Millossovich (2011), Asimit et al. (2013) and Cai et al. (2014) study optimal reinsurance contracts with default risk if there is one policyholder. We differ by allowing for multiple policyholders. In case there are multiple policyholders, the issue to allocate the remaining assets in default exists naturally. Pooling risk of multiple policyholders reduces the probability of default. This pooling should therefore be reflected in the insurance price. Bauer and Zanjani (2015), Ibragimov et al. (2010), Laux and Muermann (2010) and Sherris (2006) all assume a proportional bankruptcy rule. An exception is Araujo and Páscoa (2002), who focus on existence of general equilibria with a continuum of policyholders. There are frequent real life deviations from the proportional rule and some are actually contemplated by law in the form of seniority criteria. For instance, some rules lead to priority for some claimants such as tax authorities, employees, secured creditors and some among the unsecured creditors (Araujo and Páscoa, 2002).

This paper extends the approach of Mahul and Wright (2004) to the setting of equilibria in case the insurer is protected with limited liability. Mahul and Wright (2004) study optimal risk sharing among insurers via pools in the context of catastrophe insurance. Their perspective is to maximize a weighted utility of all insurers. Then, all insurance risk is pooled *ex post*, and then redistributed among the insurers. The premium may be decided *ex post* as well. This problem is in line with classical Pareto optimal risk sharing as in Borch (1962), but with a participation constraint. Mahul and Wright (2004) describe the constrained equal loss (CEL) recovery rule and characterize it via an *ex post* participation constraint. Our focus is different as we study the effect of rules to allocate default losses, and their effects on insurance premia and the risk taking behavior of the insurer. We assume that the premia are paid *ex ante* and determined by the insurer. Moreover, we do not optimize the joint utility of all agents, but the insurer optimizes its utility under participation constraints of the policyholders.

This paper is set out as follows. Section 2 defines the model set-up. Section 3 characterizes the optimal pooling and recovery rules. Section 4 shows existence of the equilibrium. Section 5 studies risk shifting. Section 6 shows in a numerical illustration the welfare losses of suboptimal recovery rules and the effect of the number of policyholders. Section 7 illustrates the effect of dead-weight costs in default. Finally, Section 8 concludes.

2 Preferences

2.1 Preferences insurer

We consider a one-period economy with a given future reference period. The insurer has initial wealth $W \geq 0$. Let $N = \{1, \ldots, n\}$ be the finite set of policyholders with $n \geq 1$. Every policyholder $i \in N$ pays a single premium given by $\pi \geq 0$ to insure its risk $X_i \in L^1$ with the insurer, where L^1 is the set of non-negative random variables on a given probability space for which the expectation exists. The insurer cannot observe heterogeneity in the risks that policyholders withhold, and therefore charges the same premium to everyone. Denote the set of risks as $X := (X_i)_{i=1}^n$. The insurer can invest a fraction $\alpha \in [0, 1]$ of its wealth in a risky technology that generates a stochastic excess return R, for which the support is a subset of $[-1, \infty]$. The risk-free rate is given by $r^f \geq 0$.

Before covering the insurance claims, the assets of the insurer at the given future time are given by

$$A(\alpha, \pi) = (W + n\pi)\,(1 + r^f + \alpha R),$$

which is stochastic at time 0. The insurer remains solvent if the assets are higher than the realized insurance claims, i.e., when the following event occurs:

$$S(\alpha, \pi) := \left\{ A(\alpha, \pi) \geq \sum_{i=1}^{n} X_i \right\}.$$

There are no costs of bankruptcy included for the insurer, but the policyholders are cut in their indemnities to cover the deficits. The objective of the insurer is to maximize $E\left[(A(\alpha, \pi) - \sum_{i=1}^{n} X_i)^+\right]$ under participation constraints of the policyholders, which we will specify in Subsection 2.2.

2.2 Preferences policyholders and bankruptcy rules

In this paper, we study the effects of limited liability. In case of default, the remaining assets are allocated to the policyholders. The way this should be done is non-trivial, and a central topic of this paper.

We focus on a mapping $f : L^1 \times (L^1)^n \to (L^1)^n$ that maps every stochastic realization of the risks in a allocation. It satisfies

$$\sum_{i=1}^{n} f_i\,(A, X) = \begin{cases} \delta A, & \text{if } A < \sum_{i=1}^{n} X_i, \\ \sum_{i=1}^{n} X_i, & \text{otherwise,} \end{cases}$$

and $f_i\,(A, X) \leq \delta X_i$ for all $i \in N$, where the fraction $\delta \in (0, 1]$ reflects the costs of default that are deducted from the remaining assets as in Biffis and Millossovich (2011). Hence, $f(A, X)$ is an n-dimensional set of stochastic variables that represent the posterior payments to the n policyholders. In this section, we set $\delta = 1$ and ignore deadweight costs of default. We will discuss the effect of the bankruptcy cost δ in more detail in Section 7.

Definition 2.1 *Let $\mathcal{F}$ the collection of the mappings $f : L^1 \times (L^1)^n \to (L^1)^n$ that are continuous in the first argument (assets A), and such that $\sum_{i=1}^{n} f_i(A, X) = \min\{A, \sum_{i=1}^{n} X_i\}$ and $f_i\,(A, X) \leq X_i$ for all $i \in N$. Moreover, let $RR \subset \mathcal{F}$ the collection of mappings f that are also such that $f(A, X) \geq 0$ for all $(A, X) \in L^1 \times (L^1)^n$.*

We assume that the rule f is common knowledge before the insurance contract is sold. Therefore, it might influence the insurance premium in equilibrium. In the literature, authors have studied the effect of limited liability on insurance premia. One typically focuses typically on liability with one policyholder (Cai et al., 2014; Filipović et al., 2015). Alternatively, some authors study insurance with limited liability for a given proportional recovery rule (Sherris, 2006; Ibragimov et al., 2010; Laux and Muermann, 2010).

We model the preferences of the policyholders by agents with expected utility function u and initial wealth w_0, i.e., the utility of policyholder i is given by

$$E\left[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))\right].$$

In line with Filipović et al. (2015), we impose the following regularity assumptions.

Assumption 2.1: Throughout this paper, we impose the following assumptions:

- (R, X) has a compact support, which is a subset of $[-1, \infty) \times \mathbb{R}_+^n$, and admits a jointly continuous density function. Moreover, it holds that $E[R] > 0$.
- the no-default event $S(\alpha, \pi)$ happens with positive probability for all $(\alpha, \pi) \in [0, 1] \times \mathbb{R}_{++}$, and R is non-negatively correlated with $S(\alpha, \pi)$.
- the utility function $u : \mathbb{R} \to \mathbb{R}$ is such that $u'(\cdot) > 0$, $u''(\cdot) < 0$, $\lim_{x \to -\infty} u(x) = -\infty$ and $\lim_{x \to -\infty} u'(x) = \infty$.

- u and the density of (R, X) are such that the utility of the policyholders and the insurer are real-valued and differentiable in some neighborhood of the domain $[0,1] \times \mathbb{R}_+$ of (α, π).

The policyholders' individual rationality constraints are given by

$$E\left[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))\right] \geq \underline{u}, \tag{1}$$

for all $i \in N$, where $\underline{u} \leq E\left[u(w_0 - \pi^* - X_j + f_j^*(A(\alpha^*, \pi^*), X))\right]$ for some (f^*, α^*, π^*) and all j.

The effect of the rule f is key in the participation constraint (1). As the participation constraint (1) ensures individual rationality, we maximize the expected profit of the insurer $E\left[(A(\alpha, \pi) - \sum_{i=1}^{n} X_i)^+\right]$ under this constraint. Possible sharing of welfare gains is possible by determining the value of $\underline{u}$. Interestingly, the utility of the policyholders in (1) is not necessarily decreasing in the premium π.

3 Optimal pooling and recovery rules

3.1 Problem statement

Every policyholder wants to insure its risk given by $X_i = Y_i + Z, i \in N$, where $X \in (L^1)^n$, and $Y_i, i \in N$, are independent and identically distributed (i.i.d.). This resembles a common shock model, where the risk Z is a common shock that effects all insurance claims (see, e.g., Marshall and Olkin, 1967; Promislow, 2006). A tuple (f, α, π) is called an equilibrium if it yields the highest expected profit for the insurer, provided that the participation constraint is satisfied. This optimization problem writes as follows:

$$\max_{f,\alpha,\pi} \; E[((W + n\pi)(1 + r^f + \alpha R) - \sum_{i=1}^{n} X_i)^+], \tag{2}$$

$$\text{s.t. } E[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))] \geq \underline{u}, \text{ for all } i \in N, \tag{3}$$

$$f \in \hat{F}, (\alpha, \pi) \in [0,1] \times \mathbb{R}_+, \tag{4}$$

where $\hat{F} \in \{\mathcal{F}, RR\}$ and $\underline{u} \leq E\left[u(w_0 - \pi^* - X_i + f_i^*(A(\alpha^*, \pi^*), X))\right]$ for some given $(f^*, \alpha^*, \pi^*) \in \hat{F} \times [0,1] \times \mathbb{R}_+$ and all i. So, we might set $\underline{u}$ at the utility level in the status quo, i.e., equal to $E\left[u(w_0 - X_i)\right]$. For now, we assume that the problem in (2)-(4) has a solution. We will later show existence of this solution formally in Theorem 4.2.

In the following lemma, we show the qualitative behavior of the preferences of the insurer.

Lemma 3.1 *For all* $(\alpha, \pi) \in (0,1) \times \mathbb{R}_{++}$*, we have*

$$\frac{\partial}{\partial \alpha} E\left[(A(\alpha,\pi) - \sum_{i=1}^{n} X_i)^+\right] > 0,$$

and for all $(\alpha, \pi) \in [0,1] \times \mathbb{R}_{++}$*, we have*

$$\frac{\partial}{\partial \pi} E\left[(A(\alpha,\pi) - \sum_{i=1}^{n} X_i)^+\right] > 0.$$

Proof Let $(\alpha, \pi) \in [0,1] \times \mathbb{R}_{++}$. We get

$$\begin{aligned} \frac{\partial}{\partial \pi} E\left[(A(\alpha,\pi) - \sum_{i=1}^{n} X_i)^+\right] &= E[n(1 + r^f + \alpha R)1_{S(\alpha,\pi)}] \\ &= n(1 + r^f + \alpha E[R|S(\alpha,\pi)])\mathbb{P}(S(\alpha,\pi)) > 0, \end{aligned}$$

where the inequality follows the assumption that $\mathbb{P}(S(\alpha,\pi)) > 0$, and from the fact that (R, X) admits a jointly continuous density function implying $\mathbb{P}(R > -1|S(\alpha,\pi)) > 0$ for any (α,π).

Moreover, we get for any $(\alpha,\pi) \in (0,1) \times \mathbb{R}_{++}$ that:

$$\begin{aligned} \frac{\partial}{\partial \alpha} E\left[(A(\alpha,\pi) - \sum_{i=1}^{n} X_i)^+\right] &= E[(W + n\pi)R1_{S(\alpha,\pi)}] \\ &= (W + n\pi)E[R1_{S(\alpha,\pi)}] \\ &= (W + n\pi)(E[R]\mathbb{P}(S(\alpha,\pi)) + cov\{R, 1_{S(\alpha,\pi)}\}) \\ &\geq (W + n\pi)E[R]\mathbb{P}(S(\alpha,\pi)) > 0, \end{aligned}$$

which is due to the fact that R and $S(\alpha,\pi)$ are non-negatively correlated, $E[R] > 0$, and $\mathbb{P}(S(\alpha,\pi)) > 0$. This concludes the proof.

From Lemma 3.1, we get that for a fixed $\alpha \in [0,1]$ the utility of the insurer is strictly increasing in the premium π. For a given premium π, we get from $E[R] > 0$ and the risk-loving preferences of the insurer that the utility of the insurer is strictly increasing in α.

3.2 Optimal pooling

In this subsection, we consider the case that $\hat{F} = \mathcal{F}$, i.e., the case where we allow that $f_i(A(\alpha,\pi), X) < 0$. Then, an insurer in default can force policyholders with small realized losses to sponsor the policyholders with large losses. This mechanism is for instance enforced by a protection fund, that charges levies in case of default. In an optimal insurance contract, the total risk at default is pooled and, then, pro rata shared among policyholders such that the risk $X_i - f_i(A(\alpha,\pi), X)$ is the same for every policyholder i. We call this solution perfect pooling (PP).

Proposition 3.2 *Suppose $\hat{F} = \mathcal{F}$, i.e., the rule f is not constrained to be non-negative. Then, for all equilibrium solutions (f, α, π) to (2)-(4), we get that $f = PP$, where*

$$PP_i(A(\alpha,\pi), X) = \begin{cases} X_i + (A(\alpha,\pi) - \sum_{j=1}^n X_j)/n & \text{if } A(\alpha,\pi) < \sum_{j=1}^n X_j, \\ X_i & \text{otherwise,} \end{cases} \tag{5}$$

for all $i \in N$.

Proof It holds by construction that $\sum_{i=1}^n PP_i(A(\alpha,\pi), X) = \min\{A(\alpha,\pi), \sum_{i=1}^n X_i\}$ and $PP_i(A(\alpha,\pi), X) \leq X_i$ for all i, and so we have $PP \in \mathcal{F}$. Fix α and π. Let $f \in \mathcal{F}$. We take a Taylor expansion of u around $\hat{W} := w - \pi - (\sum_{j=1}^n X_j - A(\alpha,\pi))^+/n$ to the second order:

$$\begin{aligned} u(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X)) &= u(\hat{W}) + u'(\hat{W})(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X) - \hat{W}) \\ &\quad + \frac{1}{2}u''(\zeta_i)(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X) - \hat{W})^2, \end{aligned}$$

where ζ_i is in between $w_0 - \pi - X_i + f_i(A(\alpha,\pi), X)$ and $\hat{W}$. Clearly, it holds that $\sum_{i=1}^n (w_0 - \pi - X_i + f_i(A(\alpha,\pi), X) - \hat{W}) = 0$, and so the second term vanishes. Therefore, we get by summing over all $i \in N$ and taking the expectation that

$$\begin{aligned} &\sum_{i=1}^n E[u(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X))] \\ &= nE[u(\hat{W})] + \frac{1}{2}\sum_{i=1}^n E[u''(\zeta_i)(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X) - \hat{W})^2 \\ &\leq nE[u(\hat{W})], \end{aligned}$$

which is due to $u''(\cdot) < 0$. If $f_i \neq PP_i := X_i - (\sum_{j=1}^n X_j - A(\alpha,\pi))^+/n$ for some $i \in N$, we get a strict inequality. Hence, PP, which is defined in (5), uniquely solves the following the system

$$\max_{f \in \mathcal{F}} \sum_{i=1}^n E\left[u(w_0 - \pi - X_i + f_i(A(\alpha,\pi), X))\right], \tag{6}$$

$$\text{s t.} \quad \sum_{i=1}^n f_i(A(\alpha,\pi), X) = \min\left\{A(\alpha,\pi), \sum_{i=1}^n X_i\right\}. \tag{7}$$

Suppose that $f^*(A(\alpha,\pi),X)$ is an optimal rule such that $f^* \neq PP$. Since PP solves (6)-(7) uniquely, we get that there exists a policyholder $i \in N$ such that

$$E[u(w_0 - \pi - X_i + \hat{f}_i(A(\alpha,\pi),X))] > E[u(w_0 - \pi - X_i + f_i^*(A(\alpha,\pi),X))].$$

Then, we have for this policyholder i that

$$\begin{aligned} E[u(w_0 - \pi - X_i + \hat{f}_i(A(\alpha,\pi),X))] &> E[u(w_0 - \pi - X_i + f_i^*(A(\alpha,\pi),X))] \\ &\geq \underline{u}. \end{aligned}$$

Since the utility level $E\left[u(w_0 - \pi - X_i + \hat{f}_i(A(\alpha,\pi),X))\right]$ is the same for every policyholder i, we get that if $f^*(A(\alpha,\pi),X)$ is optimal, then the participation constraint in (4) is slack. Since the utility of the policyholder is continuous in π, there exists a premium $\hat{\pi} > \pi$ such that the participation constraint in (4) is still satisfied. Since the utility of the insurer is strictly increasing in the price π, we get a higher utility for the insurer. This is a contradiction with the assumption that f^* is optimal. Hence, $f^* = PP$ is the unique rule for all solutions (f^*,α^*,π^*) to the problem (2)-(3). This concludes the proof.

The solution PP in (5) can be seen as perfect risk pooling. To enforce this egalitarian mechanism, some policyholders might need to pay after the risk occurs (*ex post*). This happens when $f_i(A,X) < 0$. This is typically difficult to enforce, as it requires policyholders to pay a compensation on top of their indemnity after the indemnities are realized. This is why we focus in the next section on the case where we impose the condition $f(A,X) \geq 0$. So, such a rule is *ex ante* not necessarily optimal. However, in practice, it is difficult to enforce cross-payments among policyholders at a future time period.

3.3 Optimal recovery rules

There is substantial literature on bankruptcy problems, which are also called rationing problems. In a standard bankruptcy problem, there is one deterministic estate $E > 0$ and a deterministic claim vector $d \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n d_i > E$ (see, e.g., O'Neill, 1982, or the overviews of Moulin, 2002, and Thomson, 2003). A bankruptcy rule φ is such that $\varphi(E,d) \geq 0$ and $\sum_{i=1}^n \varphi_i(E,d) = E$. In this paper, we generalize the concept of bankruptcy rules to allow for stochastic estate and claims, and moreover, it is well-defined in case of no default. We apply bankruptcy rules in insurance with limited liability, and we call such a rule a *recovery rule*.

We focus on the following recovery rules that are inspired by well-known bankruptcy rules (Moulin, 2000, 2002; Thomson, 2003):

- Proportional rule: for each (A, X),

$$f_i(A, X) = PROP_i(A, X) = \min\left\{\frac{A}{\sum_{j=1}^n X_j} \cdot X_i, X_i\right\},$$

  for all $i \in N$.

- Constraint Equal Award: for each (A, X), $f_i(A, X) = CEA_i(A, X) = \min\{X_i, \gamma\}$, where $\gamma \leq \max_j X_j$ is such that $\sum_{j=1}^n \min\{X_j, \gamma\} = \min\{A, \sum_{j=1}^n X_j\}$.

- Constraint Equal Loss: for each (A, X), $f_i(A, X) = CEL_i(A, X) = \max\{0, X_i - \gamma\}$, where $\gamma \leq \max_j X_j$ is such that $\sum_{j=1}^n \max\{0, X_j - \gamma\} = \min\{A, \sum_{j=1}^n X_j\}$.

- Talmud rule:

$$f(A, X) = TR(A, X) = \begin{cases} X & \text{if } \sum_{i=1}^n X_i \leq A, \\ CEA(A, \frac{1}{2}X) & \text{if } \sum_{i=1}^n X_i \geq 2A, \\ X - CEA(\sum_{i=1}^n X_i - A, \frac{1}{2}X), & \text{otherwise,} \end{cases}$$

  for each (A, X).

The intuition of the first three recovery rules is straightforward. Proportional recovery rules seem the most natural way to allocate assets in default, and is popular in the insurance literature (Sherris, 2006; Ibragimov et al., 2010; Laux and Muermann, 2010). It is easy to communicate to the policyholders. The constrained equal award rule strives to obtain egalitarianism in rationing problems (see, e.g., Koster and Boonen, 2014). The constrained equal loss rule strives to obtain egalitarianism for the dual problem, i.e., for the risks $X_i - f_i(A, X), i \in N$. In fact, Young (1988) shows for rationing problems that CEL and CEA are dual of each other, whereas the proportional rule is self-dual. The Talmud rule is more advanced, and is characterized via a consistency axiom by Aumann and Maschler (1985).

Note that some recovery rules might yield the same posterior joint risk $f(A, X)$. For instance, if $X_i = Z$ for all $i \in N$, we have that all recovery rules defined above yield the same solution, which is $f_i(A, X) = \min\{\frac{A}{n}, Z\}$ for all i.

Theorem 3.3 *For every solution (f, α, π) of* (2)-(4) *with $\hat{F} = RR$, we get that*

$$f(A(\alpha, \pi), X) = CEL(A(\alpha, \pi), X).$$

Proof Fix π and α, and moreover fix a realization $R = r$ and $X = (x_1, \ldots, x_n)$. Then, if $(W + n\pi)(1 + r^f + \alpha r) \geq \sum_{i=1}^n x_i$, then the recovery rule $f \in RR$ is fixed. So, let $\hat{A} := (W + n\pi)(1 + r^f + \alpha r) < \sum_{i=1}^n x_i$. Then, CEL is analogous to deductible insurance. So, the problem writes as

$$\max_{b_1,\ldots,b_n} \sum_{i=1}^n u(w_0 - \pi - x_i + b_i), \tag{8}$$

$$\text{s t.} \quad b_i \geq 0, \tag{9}$$

$$\sum_{i=1}^n b_i = \hat{A}. \tag{10}$$

The objective function in (8) is concave and the constraints are affine. Hence, we get all solutions from the Karush-Kuhn-Tucker (KKT) conditions:

$$u'(w_0 - \pi - x_i + b_i) + \gamma_i = \lambda, \text{ for all } i \in N,$$

where $\gamma_i b_i = 0$ and $\gamma_i \geq 0$. Since $u_i''(\cdot) < 0$, we have that all $x_i + b_i$ is the same for all policyholders whenever $\gamma_i = 0$. If $\gamma_i > 0$, then $b_i = 0$ and $u'(w_0 - \pi - x_i) < \gamma$. So, if $b_i = 0$, the utility of policyholder i is higher than the utility of policyholder j with $b_i > 0$. Therefore, we directly get that $b_i = CEL_i(\hat{A}, (x_i)_{i=1}^n), i \in N$ is the unique solution of (8)-(10). Hence, when we solve (8)-(10) for any realization of (R, X), we get that $CEL_i(A(\alpha, \pi), X)$ solves uniquely the problem

$$\begin{aligned}
&\max_f \sum_{i=1}^n E\left[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))\right], \\
&\text{s t.} \quad f(A(\alpha, \pi), X) \geq 0, \\
&\qquad \sum_{i=1}^n f_i(A(\alpha, \pi), X) = \min\left\{A(\alpha, \pi), \sum_{i=1}^n X_i\right\}.
\end{aligned}$$

Suppose that $f^*(A(\alpha, \pi), X)$ is an optimal recovery rule. Since CEL solves (8)-(10) uniquely, we get that there exists a policyholder i such that

$$\begin{aligned}
&E[u(w_0 - \pi - X_i + CEL_i(A(\alpha, \pi), X))] \\
&\qquad > E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))].
\end{aligned}$$

Then, we have for this policyholder i that

$$\begin{aligned}
E[u(w_0 - \pi - X_i + CEL_i(A(\alpha, \pi), X))] &> E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))] \\
&= \underline{u},
\end{aligned}$$

where CEL is a recovery rule as well. Note that the *ex ante* expected utility level $E\left[u(w_0 - \pi - X_i + CEL_i(A(\alpha, \pi), X))\right]$ is the same for every policyholder i. So, we get that if $f^*(A(\alpha, \pi), X)$ is an optimal recovery rule, then the participation constraint in (4) is slack. Since the utility of the policyholder is continuous in π, there exists a premium $\hat{\pi} > \pi$ such that the participation constraint (4) is still satisfied. Since the utility of the insurer is strictly increasing in the premium π, we get a higher utility for the insurer. This is a contradiction with the assumption that the recovery rule f^* is optimal. Hence, CEL is the unique solution to (6)-(7). This concludes the proof.

If f is the Constrained Equal Loss rule, then the participation constraint in (1) writes as

$$E\left[u(w_0 - \pi - \min\{X_i, \gamma\})\right] \geq \underline{u}, \tag{11}$$

where γ is a random variable such that

$$\sum_{i=1}^{n} \min\{X_i, \gamma\} = \left(\sum_{i=1}^{n} X_i - (W + n\pi)(1 + R)\right)^+ .$$

Hence, the Constrained Equal Loss recovery rule resembles deductible insurance, but where the deductible is random as well.

4 Existence of the equilibrium

In order to show existence of the equilibrium, we first show convexity of the utility of the insurer, and concavity of the utility of the policyholder, both with respect to α and π. This result holds straightforward in case $n = 1$ (see Filipović et al., 2015), but concavity of the utility function of the policyholder is more complicated to show in case $n > 1$. We assert this result in the following lemma.

Lemma 4.1 *Let $f \in \{PP, CEL\}$. The utility of the insurer is convex in α and π and the utility of the policyholders is concave in α and strictly concave in π.*

Proof Let $f = CEL$. For fixed $R = r$ and $X = (x_1, \ldots, x_n)$, the function $((W + n\pi)(1 + r^f + \alpha r) - \sum_{i=1}^{n} x_i)^+$ is convex in π and in α. Taking expectation preserves these properties. Hence, the utility of the insurer is convex in α and π

Next, we show strict concavity of the utility of the policyholder with respect to premium π. Let $0 \leq \pi_1 < \pi_2$, $\alpha \in [0,1]$, and $\lambda \in (0,1)$. Then, we get

$$\begin{aligned}
&\sum_{i=1}^{n} [\lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X)] \\
&= \lambda \min\{A(\alpha,\pi_1), \sum_{i=1}^{n} X_i\} + (1-\lambda)\min\{A(\alpha,\pi_2), \sum_{i=1}^{n} X_i\} \\
&\leq \min\{A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), \sum_{i=1}^{n} X_i\}, \qquad (12)
\end{aligned}$$

which holds due to $\lambda A(\alpha,\pi_1) + (1-\lambda)A(\alpha,\pi_2) = A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2)$. Moreover, we get

$$0 \leq \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X) \leq X_i \text{ for all } i. \qquad (13)$$

Moreover, we get from Theorem 3.3 that there exists a policyholder i such that

$$\begin{aligned}
&E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + CEL_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))] \qquad (14)\\
&\geq E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + f_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))], \qquad (15)
\end{aligned}$$

for all $f \in RR$. Hence, from (12)-(13), we get that this also holds for $\hat{f}_i = \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X), i \in N$ due to the assumption that u is increasing. Since $\hat{f}$ also yields the same *ex ante* expected utility for all policyholders, we get

$$\begin{aligned}
&E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + CEL_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))] \\
&\geq E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X))] \\
&> \lambda E[u(w_0 - \pi_1 - X_i + CEL_i(A(\alpha,\pi_1),X))] \\
&\quad + (1-\lambda)E[u(w_0 - \pi_2 - X_i + CEL_i(A(\alpha,\pi_2),X))].
\end{aligned}$$

Here, the last inequality follows from strict concavity of u, and the fact that from $\pi_1 < \pi_2$, $S(\alpha,\pi_2) \geq S(\alpha,\pi_1)$, and $\mathbb{P}(S(\alpha,\pi_1)) > 0$ it follows that $-\pi_1 - X_i + CEL_i(A(\alpha,\pi_1),X) \overset{d}{\neq} -\pi_2 - X_i + CEL_i(A(\alpha,\pi_2),X)$. Hence, the utility of the policyholder is strictly concave in π.

Showing concavity of the utility of the policyholder with respect to parameter α is analogous to the proof of concavity with respect to the premium π.

The proof for when $f = PP$ is similar, but easier, since for any fixed $R = r$ and $X = (x_1, \ldots, x_n)$ we get that $u(w_0 - \pi - (\sum_{i=1}^{n} x_i - (W + n\pi)(1 + r^f + \alpha r))^+/n)$ is concave in α and π.

Lemma 4.1 illustrates the qualitative behavior of utility functions of the insurer and policyholder, but also helps us to show existence of an equilibrium satisfying (2)-(4). This is asserted by the following theorem.

Theorem 4.2 *Let $\hat{F} \in \{\mathcal{F}, RR\}$. For any reservation utility level $\underline{u}$, there exists at least one optimal policy (f^*, α^*, π^*) that solves (2)-(4). It is such that the participation constraint in (4) is binding. Moreover, for any $\alpha^* \in [0,1]$, there exists at most one equilibrium policy (α^*, π), where π is such that (4) is binding and $\frac{\partial}{\partial \pi} E[u(w_0 - \pi + X_i + f_i(A(\alpha^*, \pi), X)) \leq 0$.*

Proof Let $\hat{F} \in \{\mathcal{F}, RR\}$. If a solution to (2)-(4) exists, we get $f^* = CEL$ if $\hat{F} = RR$ (Theorem 3.3), or $f^* = PP$ if $\hat{F} = \mathcal{F}$ (Proposition 3.2). Since the objective is strictly increasing in π, we aim for every $\alpha \in [0,1]$ to find the largest π such that the participation constraint in (4) is satisfied. If $\pi \to \infty$, we get $E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))] < E[u(w_0 - \pi)] \to -\infty$ due to $\lim_{x \to -\infty} u(x) = -\infty$. Then, the participation constraint in (3) is violated. Since the rule f^* is continuous in the assets A, the policyholder's expected utility is continuous in the premium π. Since the utility of the insurer is strictly increasing in π, we get that for any fixed $\alpha \in [0,1]$ there can be at most one optimal premium π satisfying (2)-(4). If it exists, (α, π) is such that the participation constraint in (3) is binding. By concavity of the utility of the policyholder for given α (see Lemma 4.1), it is characterized by the fact that it must also satisfy $\frac{\partial}{\partial \pi} E[u(w_0 - \pi + X_i + f_i^*(A(\alpha, \pi), X)) \leq 0$. By assumption, we have that there exists a $(f, \alpha, \pi) \in \hat{F} \times [0,1] \times \mathbb{R}_+$ with $E[u(w_0 - \pi + X_i + f_i(A(\alpha, \pi), X))] \geq \underline{u}$. From Proposition 3.2 and Theorem 3.3, it follows that this also hold for $f = PP$ when $\hat{F} = \mathcal{F}$ and for $f = CEL$ when $\hat{F} = RR$.

From $\lim_{\pi \to \infty} \max_{\alpha \in [0,1]} E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))] = -\infty$, we get that the level set $\{(\alpha, \pi) \in [0,1] \times \mathbb{R}_+ : E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))] \geq \underline{u}\}$ is a compact subset of $[0,1] \times \mathbb{R}_+$. Moreover, this set is non-empty by assumption. Since $E[(W + n\pi)(1 + r^f + \alpha R) - \sum_{i=1}^n X_i)^+]$ and $E[u(w_0 - \pi - X_i + f_i^*(A(\alpha, \pi), X))]$ are continuous on $[0,1] \times \mathbb{R}_+$, we conclude that the maximum in (2)-(4) for the respective reservation utility level $\underline{u}$, is attained in $[0,1] \times \mathbb{R}_+$.

In case of multiple policyholders, the utility for the policyholder does not need to be monotonic in the premium π, even if $R \overset{d}{=} 0$. Filipović et al. (2015) show a first-order condition to determine the equilibrium, and show uniqueness of the equilibrium. We cannot straightforwardly extend the results of Filipović et al. (2015) to our setting.

It is important to remark that if $\underline{u}$ is high, it is not rational for the insurer to offer the insurance contracts. Therefore, we need to verify ex post whether the equilibrium solving (2)-(4) is rational for the insurer; if rationality is violated there is no insurance in equilibrium.

5 Risk shifting

In this section, we study risk shifting in insurance. For instance, suppose that the premium π is such that $W + n\pi > \sum_{i=1}^{n} X_i$ for any X, the policyholder would prefer the insurer to invest completely risk-free. The policyholder is also willing to pay a higher premium to achieve this. In absence of a regulator, there is however no guarantee that the insurer will invest everything risk-free. This is called counterparty risk or risk shifting in insurance (see, e.g., Filipović et al., 2015).

Suppose the investment decision is not observed by the policyholder. After the policyholders pay the insurance premium, the insurer will invest its assets in order to maximize its own utility. Then, the risk shifting problem is given by:

$$\max_{f,\alpha,\pi} E[((W + n\pi)(1 + r^f + \alpha R) - \sum_{i=1}^{n} X_i)^+], \tag{16}$$

$$\text{s.t. } E[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))] \geq \underline{u}, \text{ for all } i \in N, \tag{17}$$

$$f \in \hat{F}, \pi > 0, \tag{18}$$

$$\alpha \in \operatorname{argmax}_{\alpha' \in [0,1]} E[((W + n\pi)(1 + \alpha' R) - \sum_{i=1}^{n} X_i)^+], \tag{19}$$

where $\hat{F} \in \{\mathcal{F}, RR\}$, and where $\underline{u} = E[u(w_0 - \pi^* - X_i + f_i(A(1, \pi^*), X))]$ for some $\pi^* > 0$. We assume that $\frac{\partial}{\partial \pi} E[u(w_0 - \pi^* - X_i + f_i(A(1, \pi^*), X))] \leq 0$. This is without loss of generality since $\lim_{\pi \to \infty} E[u(w_0 - \pi - X_i + f_i(A(1, \pi), X))] = -\infty$.

Theorem 5.1 *Let $\hat{F} \in \{\mathcal{F}, RR\}$ and the reservation utility be given by $\underline{u} = E[u(w_0 - \pi^* - X_i + f_i(A(1, \pi^*), X))]$ for some $\pi^* > 0$. Then, there exists a unique solution (f, α, π) to the problem* (16)-(19). *This is such that $(\alpha, \pi) = (1, \pi^*)$.*

Proof Let $\hat{F} \in \{\mathcal{F}, RR\}$. First, we let $\pi > 0$. From Lemma 3.1, we get for any $\alpha \in (0, 1)$ that

$$\frac{\partial}{\partial \alpha} E\left[((W + n\pi)(1 + r^f + \alpha R) - \sum_{i=1}^{n} X_i)^+\right] > 0.$$

So, since the utility of the insurer is continuous, we get that the constraint 19 yields $\alpha = 1$. Hence, the optimal solutions to (16)-(19) are such that $\alpha = 1$.

Then, the problem (16)-(19) boils down to maximize for a fixed $\alpha = 1$ the objective (16) over all $\pi \geq 0$ such that (17) is satisfied. Similar to Proposition 3.2 and Theorem 3.3, we get that the optimal rule is unique, and given by $f = PP$ if $\hat{F} = \mathcal{F}$, and $f = CEL$ if $\hat{F} = RR$. The objective function in (16) is continuous and strictly increasing in the premium $\pi \geq 0$. Moreover, by definition, there exists a $\pi^* > 0$ such that $E[u(w_0 - \pi^* - X_i + f_i(A(1, \pi^*), X))] \geq \underline{u}$, and moreover we have $\lim_{\pi \to \infty} E[u(w_0 - \pi - X_i + f_i(A(1, \pi), X))] = -\infty$. Hence, there is a unique solution, and it is such that the participation constraint is binding. Because the utility of the policyholder is strictly concave in the premium π (Lemma 4.1) and $\frac{\partial}{\partial \pi} E[u(w_0 - \pi^* - X_i + f_i(A(1, \pi^*), X))] \leq 0$, we have $\pi = \pi^*$. This concludes the proof.

6 Calibration study

In this section, we show the effect of recovery rules on equilibrium prices, and risk taking behavior of the insurer. We provide an extensive example of an insurer whose financial position is relatively poor. In that case, the effect of recovery rules are shown to be important.

Let $r^f = 0\%$, $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \exp(1)$, and $R = e^G - 1, G \sim N(\mu, \sigma^2)$, with $\mu = 0$ and $\sigma = 16\%$, and independent of X_i. Moreover, policyholders use the exponential (Constant Absolute Risk Aversion) utility function $u(x) = -\exp(\lambda x)$ with $\lambda = 0.2$. It is well-known that initial wealth w_0 is irrelevant for exponential utilities. The initial assets of the insurer are set at $W = 0$, and moreover we have $n = 10$. We set $\underline{u} = E[u(w_0 - X_i)]$. In absence of default, we get from straightforward calculations that the indifference price for insurance is approximately 1.108, i.e., the risk premium is given by 10.8%.

In the baseline model, we let $f = CEL$, but we study different recovery rules as well. For instance, and in line with Ibragimov et al. (2010), bankruptcy losses may be ex-post pro-rata shared among policyholders. We simulate the risks in the economy 100,000 times.

In general, the equilibria do not need to be unique. In this example, it turns out to be the case that the equilibrium is unique. We show the outcome on prices and risk taking in Table 1. We find that the effects of the choice of the recovery rule are substantial. For instance, when the insurer uses CEA instead of CEL, then the insurance premium will drop from 0.96 to 0.88. As a result, the probability of default increases and the expected profit for the insurer is smaller. For recovery rules, the results in Table 1 confirm Theorem 3.3 in that CEL is optimal to use for the insurer. It leads to a higher premium, and the utility for the insurer is highest. If there is perfect pooling as in Section 3.2, we find that there are additional expected profits for the insurer, but small.

Next, we show the effect of the number of policyholders, given by n. We display these effects in Table **??**. When n gets large, the total insurance losses get approximately normally distributed due to the central limit theorem. Then, default particularly occurs when investment returns are low. [TO BE COMPLETED]

f	CEL	$PROP$	CEA	TR	PP
π	0.96	0.88	0.94	0.95	0.96
α	86%	85%	100%	90%	93%
$\mathbb{P}(S(\alpha,\pi))$	49.8%	40.0%	47.5%	48.7%	49.8%
$\mathbb{E}[(A(\alpha,\pi)-\sum_{i=1}^{n} X_i)^+]$	1.21	0.83	1.15	1.17	1.24

Table 1: Overview of numerical result corresponding to Section 6. This table displays the effect of the recovery rule f. The definition of PP is provided in (5), and the other alternatives of f are shown in Subsection 3.3. This table show the equilibrium solution (α,π) of (2)-(4) with given f, and the no-default probability and the utility of the insurer in this equilibrium.

Finally, we conclude this section with analyzing the effect of the common shock. Let the common shock be given by $Z=\gamma e^{\bar{G}}$, with $\bar{G}\sim N(0,1)$ and $\gamma\in[0,1]$. Moreover, we assume $X_i=Z+(1-\gamma)Y_i$, $i=1,\dots,I$, with $Y_1,\dots,Y_n \overset{i.i.d.}{\sim} \exp(1)$. So, for every γ, the expectation is the same. The marginal distribution of R is the same as above, but $(\bar{G},G)$ are jointly normally distributed where the correlation coefficient is assumed to be -0.25. [TO BE WRITTEN]

7 Effects bankruptcy costs

In this section, we discuss the effect of bankruptcy costs δ. The preferences of the policyholder are however no longer be concave or, even, continuous. Therefore, an equilibrium does not need to exist if $\delta<1$. If equilibria exist, it is not difficult to see that Proposition 3.2 and Theorem 3.3 still hold for $\delta\in(0,1)$.

Next, we study the effect of δ numerically in cases where the equilibrium exists. We use the same setting as in Section 6, but vary δ. We display the equilibrium contracts in Table [TO BE COMPLETED], which we find to exist and to be unique.

8 Conclusion

This paper studies the effect of recovery rules in optimal insurance policies in equilibrium. We study the well-known common shock model for the joint distribution of the risks of the policyholders. We find that a *constrained equal loss* rule is optimal. This rule is popular in the field of rationing (e.g., Moulin, 2002; Thomson, 2003), but it is not commonly studied in the literature about limited liability in insurance. In the literature, proportionality is typically assumed exogenously.

If a protection fund can charge levies to policyholders with low losses, it is optimal to perfectly pool the risk. This yields the highest utility in the market. If policyholders cannot be forced to pay a levy at the future time period, we show that the constrained equal loss rule is optimal. We show existence of an equilibrium in the market, and study the welfare losses if other recovery rules are used in default. Moreover, we show that in absence of a monitoring device, the insurer will always invest all its assets in the risky technology. Therefore, the insurance price should include this risk-taking as it affects likelihood and size of a default event.

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