Valuation of Variable Long-term Care Annuities with Guaranteed Lifetime Withdrawal Benefits: A Variance Reduction Approach

Abstract

This paper proposes an efficient valuation algorithm for a variable Long-term Care Annuity with Guaranteed Lifetime Withdrawal Benefits (GLWB). This innovative product provides retirement solutions for both longevity risk and long-term care protection. The product includes the benefits of guaranteed income streams and long-term care expenses for retirees. However, the valuation of this type of product is very complicated and time-consuming. In this paper, we propose a Monte Carlo valuation algorithm that uses the variance reduction technique. The numerical results indicate that the proposed valuation algorithm is very efficient under a broad range of asset models. The proposed algorithm provides a general approach for a rapid valuation of this type of product and can help provide life insurance companies offering innovative products with an appropriate valuation tool.

Keywords: Variable Annuity; Long-term Care; Long-term Care Annuity; Guaranteed Lifetime Withdrawal Benefit; Variance Reduction

JEL Classification: G22

1. Introduction

The demand for annuities and Long-Term Care (LTC) insurance have increased with improvements in medical technology and greater awareness of longevity risk. However, the market shares of these retirement products are still limited because of the adverse selection and strict underwriting problems. Previous studies indicate that a new innovative retirement product, the so-called Long-term Care Annuity (LCA), which is a combination of a lifetime annuity and long-term care insurance, may be able to resolve the problems associated with underwriting in the long-term care insurance market while offering consumers a lower price than traditional LTC products. A number of recent studies have indicated that the LCA can help inject new life into an otherwise stagnant long-term care insurance market (Murtaugh, Spillman, and Warshawsky 2001; Brown and Warshawsky 2013; Webb 2009).

To meet the growing demand for LCA products, a re-think of the design of such products is also required. In this paper, we propose a new hybrid product that combines LCA with a variable annuity incorporating Guaranteed Lifetime Withdrawal Benefits (GLWB[1]) or Guaranteed Minimum Withdrawal Benefits (GMWB[2]). Variable Annuities with embedded guarantees have been very popular with policy-holders over the past 10 years. These investment-linked insurance products can eliminate

[1] GLWB offer a lifelong withdrawal guarantee and minimum withdrawal guarantee.
[2] GMWB offer a minimum withdrawal guarantee, and constitute a special case of GLWB when the future lifetime of the insured is deterministic.

downside risk while still providing upside retirement income potential. Faced with volatile financial markets and low interest rates, consumers look for a higher return but avoid downside risk. Compared to traditional LTC products, LCA products are better products for insurers because they offer the benefits of lower adverse selection costs and fewer underwriting problems. However, under the current low interest rate environment, LCA products are general account products and are very expensive. On the other hand, variable LCA products with GLWB have the advantage of providing upside income potential at a lower cost because they are specific account products. Therefore, a variable LCA product with GLWB is a better solution to retirement than a traditional LCA product.

However, the valuation of variable LCA products with GLWB or GMWB is very complicated. The benefits of this proposed new product depend on the subaccount value and health status of the policy-holder and are thus path-dependent. Due to the complex model of disability, the underlying option pricing problems lack a straightforward closed-form solution. The previous literature has suggested that the Monte Carlo method is the only feasible numerical method for the valuation of these LCA products (Glasserman 2004; Asmussen and Glynn 2007). However, when using the Monte Carlo method to evaluate the variable LCA product with GLWB, the risk neutral scenarios of the subaccount value are often simulated, and the Monte Carlo method has the disadvantage of a slow convergence rate and is time-consuming. Therefore, we propose a fast valuation algorithm using the techniques of variance reduction. In this paper, we further compare the efficiency of the proposed approach with the crude Monte Carlo method. As far as we know, this paper is the first study to investigate the valuation of the variable LCA with guaranteed insurance products.

Modeling the transition probability of the health state of the policy-holders is the key issue in LTC insurance, and has a huge impact on insurance premiums. Levikson and Mizrahi (1994) analyzed a general Markovian multistate model for LTC insurance contracts and developed it into an actuarial model of disability. Other studies have focused on how Markov models can be used to develop an actuarial model of disability. Czado and Rudolph (2002) estimate transition intensities using the Cox proportional hazard model which allows for the inclusion of censored observations and time-dependent risk factors with a graduated approach. Albarran et al. (2005) calculate a disability-free survival probability and the disability survival probability in a multiple-state model of disability using the Spanish population. Pritchard (2006) presents a novel methodology for using interval-censored longitudinal data by parameterizing Markov models, and estimates the costs of the LTC insurance contract. Baione and Levantesi (2014) establish a parametric model to estimate transition intensities when data are limited and only aggregated information on mortality and morbidity is available.

Health insurance in the form of long-term care is generally structured by multiple-state models which allow us to represent the evolution of a given insurance policy. Multistate models can be defined in both a time-continuous and a time-discrete context and offer a powerful tool for interpreting various practical calculation methods (Pitacco 1995, 2014). In this paper, we adopt the disability transition model with a continuous-time Markov process, which is used by Manton, Corder, and Stallard (1993), Pitacco (1995), Haberman and Pitacco (1998), Murtaugh, Spillman, and Warshawsky (2001), Czado and Rudolph (2002), Albarran et al. (2005), Pritchard (2006), Brown and Warshawsky (2013), and Baione and Levantesi (2014). We adopt classification of health states and the transition intensity matrix performed by Pritchard (2006).

The problems associated with the fair valuation of GMWB have been discussed in many studies (Bacinello 2003; Chen and Forsyth 2008; Chen, Vetzal, and Forsyth 2008; Holz, Kling, and Russ 2012; Milevsky and Salisbury 2006; Dai, Kwok, and Zong 2008; Peng, Leung, and Kwok 2012; Yang and Dai 2013; Bacinello et al. 2011). In addition, the problems associated with the fair valuation of GLWB have been discussed by Bernard (2010), Holz, Kling, and Ruß (2012), Piscopo (2009) and Piscopo and Haberman (2011). However, the previous research considers only continuous withdrawal models (that assume that either the amount of the insurance fee or the policy-holder's withdrawal behavior is continuous) to simplify the complexity of the corresponding valuation problem. In this paper, we assume a discrete withdrawal model which is closer to the actual application of GMWB and GLWB in the market.

The remainder of this paper is organized as follows. In Section 2, we describe the product details and health state models. Section 3 develops our valuation models for variable LCA products with GLWB. Section 4 discusses the Monte Carlo method and variance reduction approach. Numerical results are provided in Section 5 followed by concluding remarks in Section 6.

2. Product specifications and other models

(1) Product specifications

We now turn to describe the variable long-term care annuities with guaranteed lifetime withdrawal benefits. In order to formulate more realistic assumptions for the dynamics of the account value W_t, we consider a discrete withdrawal model. In actual practice, most GLWB contracts are annually based on a discrete withdrawal scheme with fixed management fees K and guaranteed fees αW_t. Let $S(t)$ be the net asset value (NAV) of the invested mutual fund at time t. Then the annual return of the invested mutual fund over the t-th year will be:

$$R_t = \frac{S(t)}{S(t-1)};\ t = 1,2, \ldots\ldots T$$

The initial value of the sub-account W_0 equals w_0. At the beginning of year *t* (*t = 0, 1, 2, …, T-1*), a guarantee fee (α times the value of the sub-account) and a fixed management fee *K* are withdrawn from the sub-account by the insurer. We let $M_x(t)$ be the health status of the policy-holder after time *t* when he bought the policy at age *x*. The insurer pays $\theta_t\big(M_x(t)\big)$ on behalf of the insured for the LTC benefit according to the different health states and $g_t w_0$ for the annuity benefit with corresponding inflation protection at *t*.[3] In addition, at the end of year *t* (*t = 1, 2, …, T-1*), the insured withdraws $g_t w_0 + \theta_t\big(M_x(t)\big)$ from the sub-account. When the insured dies at time *T*, the beneficiary can withdraw $g_t w_0 + \theta_t\big(M_x(t)\big)$ and the remaining amount of the sub-account.

Let W_t^- denote the account value at year t before these withdrawals and W_t^+ the account value in year *t* after these withdrawals. The process of the account value can then be expressed as

$$W_0^- = w_0, \qquad W_0^+ = \max\big((1-\alpha)W_0^- - K, 0\big)$$

$$W_t^- = R_t W_{t-1}^+, \quad t = 1, 2, \ldots, T$$

$$W_t^+ = \max\Big(0, (1-\alpha)W_t^- - K - g_t w_0 - \theta_t\big(M_x(t)\big)\Big), t = 1, 2, \ldots, T-1$$

$$W_T^+ = max(g_T w_0 + \theta_T(M_x(T)), W_T^-)$$

[3] θ_t and g_t usually increase over year *t* because of inflation protection.

To be more precise, this contract provides the following cash-flows Y_t to the policy-holder,

$$Y_t = \left(g_t w_0 + \theta_t(M_x(t)\right), \ \ t = 1, 2, \ldots, T-1;$$

$$Y_T = \max\left(\left(g_T w_0 + \theta_T(M_x(T)\right), W_T^-\right).$$

The cash-flow received at time T can be decomposed into

$$Y_T = g_T w_0 + \theta_T(M_x(T)) + \max\left(0, W_T^- \quad - \left(g_T w_0 + \theta_T(M_x(T)\right)\right)$$

The above cash-flow Y_T is decomposed into the final payment of an LTC annuity and an option-like payment. The fair value of the variable LCA with a GLWB contract is therefore the sum of the fair values of the LTC annuity and an option. We refer to the option-like payment as the LCA-GLWB option. Therefore, the above analysis reduces the problem of the valuation of the variable LCA with GLWB to that of the LCA-GLWB option and the LCA. Based on the risk-neutral valuation principle (Harrison and Kreps 1979; Harrison and Pliska 1981), the fair value of the LCA under a continuous-time Markov chains model is

$$E_Q\left(\sum_{t=0}^{\mathrm{T}} \left(g_t w_0 + \theta_t\left(M_x(t)\right)\right) / B(t) \right)$$

and the fair value of the LCA-GLWB option can be expressed as

$$E_Q\left[\frac{\max\left(0, W_T - \left(g_T w_0 + \theta_T(M_x(T)\right)\right)}{B(T)}\right]$$

where E_Q denotes the expectations under a risk neutral measure and $B(t)$ denotes the value of a money market account with an initial account value equal to 1 at time *t*. The stochastic variables used in pricing the LCA-GLWB option comprise the health state of the insured and the annual return that is calculated based on the invested mutual fund.

(2) Health state model

We adopt a continuous-time Markov model for the health state of the policy-holder based on the literature (Manton, Corder, and Stallard 1993; Pitacco 1995; Haberman and Pitacco 1998; Murtaugh, Spillman, and Warshawsky 2001; Czado and Rudolph 2002; Baione and Levantesi 2014; Brown and Warshawsky 2013). The continuous-time Markov model for the health state may be described as follows. Consider a policy-holder aged x and suppose that the individual moves independently among different health states, denoted by health state 1, health state 2…, health state *h*. Let $M_x(t)$ be the state occupied at time t by a randomly chosen individual starting at age *x*. For $0 \leq s \leq t$, let $P^x(s,t)$ be the $h \times h$ transition probability matrix with entries

$$p_{ij}^x(s,t) = \mathrm{P}\{M_x(t) = j | M_x(s) = i\},$$

for health state i, j =1, ……, h with starting age *x*. The process can be specified in terms of the transition rates:

$$q_{ij}^x(t) = \lim_{\Delta t \to 0} p_{ij}^x(t, t+\Delta t)/\Delta t\,,\ i \neq j,$$

$$q_{ii}^{x}(t) = \lim_{\Delta t \to 0} p_{ij}^{x}(t, t + \Delta t)/\Delta t\,,\ i = 1, \dots \dots, h.$$

Under the model assumptions, we can describe the transition probabilities by Kolmogorov forward and backward equations. In what follows are the Kolmogorov forward equation

$$\frac{dp_{ij}^{x}(s,t)}{dt} = \sum_{k} p_{ik}^{x}(s,t) q_{kj}^{x}(t)$$

and Kolmogorov backward equation

$$\frac{dp_{ij}^{x}(s,t)}{ds} = \sum_{k} q_{ik}^{x}(s) p_{kj}^{x}(s,t)$$

Let $Q_x(t)$ be the $h \times h$ rate matrix with entries $q_{ij}^{x}(t)$. It is well known that $q_{ij}^{x}(t) \geq 0$ for $i \neq j$ and $\sum_{j=1}^{h} q_{ij}^{x}(t) = 0$. We assume that M_x is time-homogeneity during each year, i.e., for s = 0, 1, 2, ….

$Q_x(t) = Q_x(s),\quad \text{for } 0 \leq t - s < 1$

The transition probability matrix can then be computed via rate matrix exponential

$$P^{x}(s,t) = e^{Q_x(s)t}\,,$$

where s is a non-negative integer and $0 \leq t - s < 1$

(3) Asset model

We assume that the invested mutual fund $S(t)$ follows a Levy process, which has stationary and independent increments (Asmussen and Glynn 2007). There are a few variations of the Levy process, several of which have been used to describe the dynamics of asset prices. For example, Merton (1976) introduced a jump-diffusion model for derivative pricing. The model can be described through the stochastic differential equation

$$\frac{\mathrm{d}S(\mathrm{t})}{S(\mathrm{t}^{-})} = \mu \mathrm{dt} + \sigma \mathrm{d}Z(t) + \mathrm{d}J(t),$$

where μ and σ are constants and Z is a standard Brownian motion and J is a jump process independent of Z that can be specified as

$$\mathrm{J(t)} = \sum_{\mathrm{k=1}}^{\mathrm{N(t)}} (Y_k - 1),$$

where Y_k is a random variable and $N(t)$ a counting process. Under the assumption that $S(t)$ follows jump-diffusion model, the annual returns on the invested mutual fund over each year are independent. Then we can simulate the asset price process by simulating the number of jumps, jump arrival time and jump size before putting these things together.

Pure jump processes also have been proved adequate in describing the dynamics of asset prices (Samoradnitsky and Taqqu 1994). Madan and Seneta (1990) proposed models based on gamma processes. They refer to the constructed process as a variance gamma (VG) process. Madan, Carr, and Chang (1998) used a VG process to estimate statistical and risk neutral densities using data based on the S&P500 Index and the prices of options related to this index. They observed that the statistical density is symmetric with some kurtosis, while the risk neutral density is negatively skewed with a larger kurtosis. They also found that the additional parameters in the VG process correct the pricing biases of the Black-Scholes model. The distributions of logarithmic asset returns can often be well fitted by normal inverse Gaussian (NIG) distributions. Therefore, Barndorff-Nielsen (1997) proposed an NIG process to model the dynamics of asset prices. VG and NIG processes share some similarities. Their sample paths can be obtained through a Brownian motion characterized by a random time-change. Therefore, the generation of their sample paths is not much harder than that of a Brownian motion (see Asmussen and Glynn (2007) and Glasserman (2004) .

3. Proposed Monte Carlo Methods

Since we assume that the health state follows a CTMC process, we can simulate the health state standard using a stochastic simulation algorithm (Glasserman 2004). The discrete skeleton of the health state process can be simulated as $M_x(1), M_x(2), \dots, M_x(T)$

It is clear that the process of the account value at time t depends on the entire path of R_t and $M_x(t)$. In particular,

$$W_T^- = f(R_1, R_2, \dots, R_T, M_x(1), M_x(2), \dots, M_x(T)),$$

where $f(.)$ is the function defined by recursions in Section 2. This makes the payoff of the LCA-GLWB option path-dependent. This also implies that the Monte

Carlo method is the only viable approach for the valuation of this option (Boyle, Broadie, and Glasserman 1997; Glasserman 2004). We propose efficient Monte Carlo valuation methods by using variance reduction techniques. In particular, we use the control variates technique in accelerating the speed of the Monte Carlo methods. We provide a short description of the control variates below.

Suppose that we wish to estimate $\alpha = \mathrm{E}(L)$, where L is the output of a complex stochastic process. A naïve Monte Carlo procedure would generate m independent copies of L, and produce the standard estimate

$$\alpha_{naive} = \frac{1}{m}\sum_{i=1}^{m} L_i$$

where $L_1, \ldots, L_m$ are independent copies of L. Let X be a d by 1 random vector in which each component of X is correlated with L. Let (μ, Σ) denote the mean vector and covariance matrix of X. The mean vector μ is known. Suppose that the covariance between L_i and X is c_i and $c = (c_1, \ldots, c_d)^{\mathrm{T}}$. We can define the control variates as

$$C = X - \mu.$$

It is clear that the mean vector of $C = 0$, the covariance matrix of $C = \Sigma$, and the covariance between L and C_i is c_i . Now define

$$L_C(\lambda) = L - \lambda^{\mathrm{T}} C$$

It is obvious that

$$\mathrm{E}[L_C(\lambda)] = 0$$

and

$$\mathrm{Var}[L_C(\lambda)] = \sigma_X^2 - 2\lambda^{\mathrm{T}}c + \lambda^T\Sigma\lambda$$

The minimizer of above formula is

$$\lambda^* = \Sigma^{-1}c$$

and

$$\mathrm{Var}[L_C(\lambda^*)] = \sigma_L^2 - 2(\Sigma^{-1}c)^{\mathrm{T}}c + (\Sigma^{-1}c)^{\mathrm{T}}\Sigma(\Sigma^{-1}c)$$

Hence

$$\mathrm{Var}[L_C(\lambda^*)] = \sigma_L^2 - c^T\Sigma^{-1}c^T < \sigma_X^2$$

Let $L_C^{(\mathrm{i})}(\lambda^*), i = 1, \ldots, m$ be independent copies of $L_C(\lambda^*)$. Then it is obvious that

$$\alpha_c = \frac{1}{m}\sum_{i=1}^{m} L_C^{(\mathrm{i})}(\lambda^*)$$

is a more efficient estimate for α. It is usually not possible to compute the exact value of λ^*, since Σ and c are usually unknown. However, accurate estimates of Σ and c are easy to compute from the simulation output, and therefore an accurate estimate of λ^* is also easy to obtain. The key step in applying control variates is to find suitable control variates.

In light of the payoff function of the LCA-GLWB option, we select efficient control variates as follows. First of all, we let

$$X_1 = \big((1-\alpha)w_0 - K\big)R_1$$

$$X_t = \big((1-\alpha)X_{t-1} - K - g_t w_0\big)R_t, \qquad t = 2, \dots, T$$

It is clear that $X_t = W_t$ if the account values are all positive time $s < t$. Therefore, $X_T / B(T)$ is highly correlated with the discount payoff of the LCA-GLWB option. Since $R_1, R_2 \dots\dots R_T$ and $M_x(1), M_x(2) \dots\dots M_x(T)$ are independent, this implies that the expected value of X_T can be easily computed from the above recursions. Therefore, we use

$$C_1 = \frac{X_T}{B(T)} - E\left(\frac{X_T}{B(T)}\right)$$

as our key control variate. In addition, we consider the following control variates, which are easy to compute and are also correlated with the payoff of LCA-GLWB option:

$$C_2 = \frac{S(T)}{S(0)} - \mathrm{E}\left[\frac{S(T)}{S(0)}\right] = R_1 R_2 \cdots R_T - \mathrm{E}[R_1 R_2 \cdots R_T]$$

$$C_3 = \sum_{t=1}^{T}\Big(g_t w_0 + \theta_t\big(M_x(t)\big)\Big) - \mathrm{E}\left[\sum_{t=1}^{T}\Big(g_t w_0 + \theta_t\big(M_x(t)\big)\Big)\right]$$

$$C_4 = T - \mathrm{E}[T]$$

4. Numerical Results

Based on the valuation model and variance reduction technique described in the previous sections, we present the numerical results for the estimation of the fair value of the LTC annuity (i.e., traditional LCA) and LCA-GLWB options (i.e., the variable LCA with GLWB). We test C_1, C_2, C_3, C_4 and [C_1 C_2 C_3 C_4] as different sets of control variates. To test the effectiveness of our algorithm, we apply it to the valuation problem of the LCA-GLWB option under the simple geometric Brownian motion (GBM) assumption asset models. We further test the effectiveness of the control variate sets [C_1 C_2 C_3 C_4] under different types of asset model.

We adopt the transition rate matrix estimated and graduated by Pritchard (2006). Exhibit 1 shows the classification of the health state in our numerical example. There are seven health state categories in our example, which are denoted as follows: Healthy—state 1; 1 or more IADL—state 2; 1-2 ADLs—state 3; 3-4 ADLs—state 4; 5-6 ADLs only—state 5; institutionalized—state 6; and dead—state 7.

We design our numerical example as follows. For the annuity benefit, this contract offers a lifetime guarantee annuity $g_t w_0$ while the individual is alive. The disability benefit is a certain guaranteed amount per period as a person meets 3+ ADLs. Similar to actual LTC products, the LTC benefit includes inflation protection, which increases payments by π percent per year. Disability benefits at time t amount to θ_t, which can be defined as a certain percentage c of the guaranteed amount w_0, which includes inflation protection of π percent per year. This contract provides disability benefits θ_t under our numerical results as follows:

$$\theta_t[M_x(t), \pi, c, w_0] = 0, \qquad M_x(t) = 1,2,3,7$$

$$\theta_t[M_x(t), \pi, c, w_0] = cw_0(1+\pi)^t, \quad M_x(t) = 4,5,6$$

In our numerical example, the initial account value and the guaranteed amount is \$100,000. We assume a continuously compounded risk-free rate of 4%, an inflation protection rate of LTC benefits π of 5%, an annuity benefits withdrawal rate g_t of 2% and an LTC benefits withdrawal rate c of 6%. To understand how correlation affects the variance reduction efficiency of our variance reduction algorithm, we repeat the same calculation with different starting ages of 60, 65, 70, 75 and 80 with starting health state 1.

The combination of a variable LCA with GLWB implies the difficulty associated with estimating the fair price of this product, and the distribution of this product is more complex than that of a traditional product. Based on the analysis in the previous section, the fair price of a variable LCA with GLWB can be decomposed into the fair price of an LCA-GLWB option and the fair price of an LCA. Our focus is on estimating the fair values of the LCA-GLWB options, which are vital inputs in calculating the fair value of a hybrid product combining variable LCA with GLWB.

We report the point estimates (p.e.) and standard errors (*s.e.*) for estimators with and without control variates, where the standard error is defined as $\frac{\sqrt{\frac{1}{m-1}\sum_{i=1}^{m}(Y_i-\alpha)^2}}{\sqrt{\mathrm{m}}}$. In order to observe the efficiency achieved by the control variates, we calculate the variance reduction ratio (VRR). Using the sample variance ratio, we can find a significant gain in the efficiency between the crude estimator and the estimators based on our control variates technique. In each case, we generate 1,000,000 simulation runs for the naive Monte Carlo approach and then compare the variance reduction ratio to quantify the statistical efficiency of our control variates.

We present the numerical results for the estimation of the LCA separately in exhibit 2 and exhibit 3. In exhibit 4, we present the results for different control variates and the variance reduction under the GBM process. The control variates exhibit a marked decrease in variance. The efficiency gain is more prominent when the age x is higher. Moreover, the efficiency of our variance reduction technique increases as the value of LCA-GLWB increases, which indicates that our algorithm reduces the variance of the estimator especially when the possibility of an out-of-money event taking place is remote. As is evident in exhibit 5, the estimator for the LCA-GLWB option, based on our control variates [C_1 C_2 C_3 C_4], is substantially more efficient than that estimated by crude Monte Carlo simulations for different kinds of stock process.

Exhibit 1 Health States of Disability Model

Health State	Disability Status	Benefit (Amount)
1	Health	Annuity payment ($g_t w_0$)
2	I ADLs	Annuity payment ($g_t w_0$)
3	1-2 ADLs	Annuity payment ($g_t w_0$)
4	3-4 ADLs	Annuity payment($g_t w_0$)+ disability payment(θ_t)
5	5-6 ADLs	Annuity payment($g_t w_0$)+ disability payment(θ_t)
6	Institutionalized	Annuity payment($g_t w_0$)+ disability payment(θ_t)
7	Dead	0

Exhibit 2 The Actual Value of a Long-term Care Annuity

Entry age(x)	State at the Start of Contract (i): State 1 Health		
	LTC	Annuity	Sum
60	7442	42458	57342
65	7021	32868	46909
70	6769	25811	39350
75	6613	20472	33699
80	6506	16315	29326

Note: Assuming W0=100,000, r=0.04, g=0.02, c=0.06, π *=0.05,*

Exhibit 3 The Fair Value of a Long-term Care Annuity

Entry age(x)	State at the Start of Contract (i): State 1 Health	
	p.e.	*s.e.*
60	56956	394
65	47024	370
70	39367	337
75	33874	306
80	29312	279

Note: Assuming W0=100,000, r=0.04, g=0.02, c=0.06, π *=0.05, and the number of replicates=* 10^6.

Exhibit 4 Variance Reduction Ratio of the LCA-GLWB Option with Different Control Variates under a GBM process

Entry age(x)	Health State at the Start of Contract (i) : State 1 Health											
	Naïve			Control Variates :C1			Control Variates :C2			Control Variates :C3		
	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR
60	49680	57		49702	16	12.39	49710	44	1.71	49687	54	1.13
65	55660	54		55752	17	10.66	55690	43	1.56	55685	50	1.19
70	60458	51		60512	17	9.33	60498	42	1.49	60450	46	1.24
75	64346	48		64337	17	8.32	64308	40	1.47	64381	42	1.28
80	67701	45		67653	16	7.61	67688	37	1.47	67670	39	1.33
Entry age(x)	Control Variates :C4			Control Variates : [C1, C2]			Control Variates : [C1, C2, C3]			Control Variates : [C1, C2, C3, C4]		
	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR	p.e.	*s.e.*	VRR
60	49682	54	1.11	49696	15	15.05	49703	12	22.62	49701	11	26.70
65	55690	50	1.15	55757	15	13.57	55763	11	22.30	55758	10	27.32
70	60456	47	1.19	60505	14	12.46	60504	11	22.78	60501	10	28.74
75	64385	44	1.22	64354	14	11.83	64360	10	23.94	64356	9	31.28
80	67683	40	1.25	67648	13	11.83	67637	9	26.25	67636	8	36.01

Note: Assuming W0=100,000, r=0.04, g=0.02, c=0.06, π *=0.05, sigma of GBM=0.16, K=300,* α*=0.008 and the number of replicates=* 10^6.

Exhibit 5 Variance Reduction Ratio of Control Variates under Different Types of Stock Process

		Naïve	Control Variates						
			C1	C2	C3	C4	[C1, C2]	[C1, C2, C3]	[C1, C2, C3, C4]
GBM	A	49642	49677	49653	49642	49645	49679	49677	49678
Process [a.]	s.e.	57	16	44	54	54	15	12	11
	VRR	—	12.32	1.70	1.13	1.11	15.00	22.61	26.76
Merton	A	49700	49727	49771	49648	49677	49711	49695	49687
Process [b]	s.e.	58	16	44	54	55	15	12	11
	VRR	—	12.56	1.71	1.13	1.11	15.17	22.85	26.84
VG	A	49790	49721	49749	49777	49793	49723	49717	49713
Process [c]	s.e.	58	16	44	55	55	15	12	11
	VRR	—	12.81	1.73	1.12	1.10	15.48	23.18	27.15
NIG	A	49708	49729	49727	49707	49730	49726	49727	49715
Process [d]	s.e.	60	16	45	57	58	15	12	11
	VRR	—	13.74	1.80	1.11	1.10	16.49	24.56	28.70

Note: Assuming W0=100,000, r=0.04, g=0.02, c=0.06, π*=0.05, sigma of GBM=0.16, K=300, α=0.008 and the number of replicates=* 10^6.

[a.] *Assuming the mean is 0.04 and standard deviation is 0.16 is under GBM process.*

[b.] *Assuming the mean is 0.04 and standard deviation is 0.16, where sigma is 0.1, a=0, b=.10197, and lambda=1.5, under a Merton process.*

[c.] *Assuming the mean is 0.04 and standard deviation is 0.16, where C is 2, G=12.5 and, M=12.5, under a variance gamma (VG) process.*

[d.] *Assuming the mean is 0.04 and standard deviation is 0.16, where alpha=5, beta=0, and delta=0.128, under a normal inverse Gaussian (NIG) process.*

5. Conclusion

In this paper, we propose a new hybrid product combining long-term care benefits and a variable annuity with GLWB. We believe the variable LCA with GLWB is a better solution for retirement than the traditional LCA products since it includes the benefits of guaranteed income streams and long-term care expenses for the retirees. This innovative product should be more attractive to consumers because it has the advantages of providing upside income potential and lower cost. However, the valuation of this type of product is very complicated and time-consuming. As far as we know, this paper is the first study to investigate the valuation of the variable LCA with guaranteed insurance products.

For the valuation of the variable LCA with GLWB, we can use the Monte Carlo simulation for the path-dependent option. In order to improve the efficiency of simulation, we propose an efficient valuation algorithm for the variable LCA with GLWB by using the control variates technique. We select a set of efficient control variates. The numerical results show that the proposed valuation algorithm is very efficient and time-saving. The efficiency gain is more prominent when the age is higher. Moreover, the efficiency of the variance reduction technique increases as the value of LCA-GLWB increases. We also find that the fast Monte Carlo algorithm that we propose can be applied to very general asset models and contract designs. Therefore, our proposed algorithm provides a better way valuing and pricing products and can help life insurance companies offer this innovative retirement product more efficiently.

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