The Effect of Labor Income and Health Uncertainty on the Valuation of Guaranteed Minimum Death Benefits

1 Introduction

Guaranteed Minimum Benefits (GMxBs) embedded in Variable Annuity (VA) Contracts have been a subject of significant interest since they became popular in the 1990s. Valuation of these contracts can be done in two distinct ways following two different rationales. GMxB values are most commonly computed in a risk-neutral framework which assumes complete markets (see, for example, Bauer et al 2008). This approach makes sense when mortality risk is diversifiable and market risk can be easily hedged. This method of valuation is therefore most applicable to the insurance company issuing the policy. More recent work had focused on the valuation of these benefits from the perspective of the policyholder. This individual is not able to diversify the mortality risk nor is he typically able to hedge the market risk at low cost. In this case, it is more reasonable to value the contract using a life-cycle model which maximizes the utility of the contract to the policyholder including bequest motives. This is the approach taken in a number of recent papers, including Gao and Ulm (2012), Bauer and Moenig (2015), Gao and Ulm (2015), and Steinorth and Mitchell (2015).

Gao and Ulm (2015) combines both approaches to find a mutually agreeable price for the Guaranteed Minimum Death Benefit (GMDB). They find that GMDBs fail to provide value to the policyholder if a term life policy is also available. In other words, there is no price on which an insurer who prices in a risk-neutral fashion and a policyholder who prices by maximizes utility over his lifetime will agree. This paper introduces uncertainty in labor income and health status to the model. In contrast to the earlier study, we find that agreeable prices exist for certain parameter combinations. In particular, very risk-averse individuals who face high unemployment risk will find that purchasing a GMDB at the risk-neutral price will increase their utility. This occurs because an unemployed individual with a low VA contract value will be unwilling to pay for term insurance, but has precommitted to pay the now small GMDB fees in exchange for the death benefit.

2 The Model

In the model, we apply a constant relative risk aversion (CRRA) type utility which has a functional form

$$u(c) = \begin{cases} \dfrac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\ ln(c), & \gamma = 1. \end{cases}$$

The insured[1] and his beneficiary are risk averse with the CRRA utility type and gain utility from consumption c. γ is the coefficient of relative risk aversion.

2.1 Objective Function

In our study, we assume a GMDB with return of premium and roll-up benefits. Furthermore, we incorporate unemployment risk and health uncertainty. This extends the analysis in Gao and Ulm (2015). In the current setting, periodic income is stochastic and the mortality probability also varies according to policyholder's health status. An individual purchases a variable annuity contract with GMDB options and makes a lump sum deposit to the variable annuity account. If the individual is employed during period t, he will receive labor income and make consumption decisions. At the same time, he will also decide whether he needs a term life policy to hedge against the risk of premature death. If his labor income

[1] We use "insured" and "policyholder" interchangeably throughout the paper.

is not sufficient (e.g. because of poor health status or health uncertainty) to support his consumption and term life policy purchase or he is unemployed, he will make a decision to withdraw from his variable annuity account. In these cases, the GMDB level will be reduced proportionally with the withdrawal ratio. For simplicity, we assume there is no other investment vehicle except the VA account and there is no interim deposit to the VA account. After the consumption, withdrawal and term policy purchase decisions, still at period t, the policyholder will decide the allocation between fixed and variable subaccounts in the VA account. If the policyholder dies at time t, the amount in the VA account, which is guaranteed by the GMDB, will be a bequest inherited by his beneficiary. In addition to the inheritance from the policyholder's VA account, the beneficiary also gets the term life policy payment F^c. Because of the existence of a bequest motive, the individual makes all his decisions to maximize the joint utility of both his beneficiary and himself. After the individual dies, the beneficiary inherits the money and maximizes her own utility by optimal allocation and withdrawal. If the insured survives until his retirement age, at the end of the policy period, he will get the entire account value and annuitize it for his retirement life.

According to the description above, the objective function can be expressed as follows,

$$
\begin{aligned}
(1) \quad \max_{\omega_t, d_t, P_t} E\Bigg[\sum_{t=1}^{T} \beta^t \big(\prod\nolimits_{i=1}^{t} \phi_i\big) u(c_t) + \beta^T \big(\prod\nolimits_{i=1}^{T} \phi_i\big) V_{T+1}(a_{T+1}) + \dots \\
\dots + \sum_{t=1}^{T} \beta^t \big(\prod\nolimits_{i=1}^{t-1} \phi_i\big)(1-\phi_t)\zeta v_B(b_t + F_t^c)\Bigg],
\end{aligned}
$$

In the objective function, the insured retires at the end of period T. ω is the percentage of risky assets (wealth held in the variable subaccount) and $1-\omega$ is the proportion of wealth held in the fixed rate subaccount. Consistent with real world policies, our model assumes $\omega \in [0, 1]$, i.e. short sales are not allowed. β is the subjective discount factor. ϕ is the survival rate. ζ denotes the strength of the bequest motive[2] and ranges from 0 to 1. If $\zeta = 0$, the insured has no bequest motive and does not leave anything to his beneficiary; if

[2] An overview of bequest motives is provided by Arrondel et al. (1997) and Masson and Pestieau (1997), De Nardi (2004) and Ameriks et al. (2011).

$\zeta = 1$, the insured is assumed to have the strongest bequest motive and treat his beneficiary like himself. V is the policyholder's value function and v is the beneficiary's value function. If the insured dies before retirement, the beneficiary will get b, which equals the greater of the account value or the guaranteed amount, plus F^c. P is the premium for the term insurance policy. y is the periodic income at time t, and d is the withdrawal amount from the VA account. The periodic consumption c is the sum of d and y.

When the policyholder purchases the variable annuity contract, the beneficiary has T_B years until retirement age. If the policyholder dies at time t, the beneficiary will receive the bequest plus term life benefit and have $T_B - t$ years until her retirement. As a rational person, the beneficiary is not assumed to withdraw the total amount (b^B) in a lump sum immediately. She will consume by making withdrawal decision optimally and will allocate the remaining amount between risky and risk-free accounts. Like the policyholder, the beneficiary will make a retirement plan by transforming the money to a lifetime payout annuity after her retirement. However, the beneficiary's investment is not assumed to be protected by the GMDB and she has no bequest motive. Therefore, the beneficiary's objective function is defined as follows,

$$(2) \quad \max_{\omega_t^B, c_t^B} E\left[\sum_{t_B=t}^{T_B} \beta^{t_B - t}\left(\prod_{i=t}^{t_B-1} \phi_i\right) u(c_t^B) + \beta^{T_B - t}\left(\prod_{i=t}^{T_B} \phi_i\right) v_B(b_{T_B+1}^B)\right].$$

With these objective functions, we can derive the Bellman equation of the insured as follows,

$$(3) \quad V_t(a_t, b_t) = \max_{\omega_t, d_t, P_t} \left\{u_t(c_t) + (1 - \phi_t)\zeta v_B(b_t + F_t^c) + \beta\phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) \mid a_t, r_t]\right\}$$

subject to

$$a_1 = b_1,$$

$$c_t \equiv y_t + d_t, \quad 0 \le d_t \le a_t,$$

$$a_{t+1} = (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))(a_t - d_t), \quad 0 \le \omega_t \le 1,$$

$$
\begin{aligned}
k_{t+1} &= k_t(1+r_f)\frac{a_{t+1}-d_{t+1}}{a_{t+1}}, \\
b_{t+1} &= max(k_{t+1}, a_{t+1}), \\
F_t^c &= \frac{P_t}{(1-\phi_t)(1+\eta)}, \quad 0 \le P_t < d_t + y_t, \\
V_{T+1}(a_{T+1}) &= \sum_{t=T+1}^{T_{max}} \beta^{t-(T+1)}(\textstyle\prod_{i=T+1}^{t-1} \phi_i)u(\bar{c}).
\end{aligned}
$$

Most notation is defined above. η is the loading factor, which is the amount added to the pure premium to cover other expenses, profit, and a margin for contingencies. The fixed subaccount grows at a fixed growth rate g and expected rate of return of the risky asset is r. The return of premium and roll-up benefit level k is reduced proportionally with the withdrawal amount d. T_{max} is the longest time an individual can survive. $\bar{c}$ is the periodic consumption after retirement, and we assume it is the level payment (a life time payout annuity) from the variable annuity account[3]. Given risk free rate r_f, $\bar{c}$ can be derived from the terminal account value as follows,

$$
\begin{aligned}
a_{T+1} &= (\omega_T(1+r_{T+1}) + (1-\omega_T)(1+r_f))(a_T - d_T) \\
&= \bar{c} \sum_{t=T+1}^{T_{max}} \textstyle\prod_{i=T+1}^{t-1} \phi_i (1+r_f)^{T+1-t}, \\
\Longrightarrow \bar{c} &= \frac{a_{T+1}}{\sum_{t=T+1}^{T_{max}} \prod_{i=T+1}^{t-1} \phi_i (1+r_f)^{T+1-t}}.
\end{aligned}
$$

The beneficiary's constant consumption level after her retirement can be derived in the same way.

2.2 2-Stage Bellman Equations

Following Hardy (2003), Bauer et al. (2008) and Gao and Ulm (2012, 2015), all state variables are denoted as $(\cdot)_{t^-}$, $(\cdot)_{t^+}$, i.e. the value immediately before and after the trans-

[3] Chen, et al. (2006) introduce life time payout annuity which converts an accumulated investment amount to a periodic payouts over the life of investors. As in Gao and Ulm (2012, 2015), we assume there is no bequest motive after the individual retires and converts his VA account value to a lifetime payout annuity.

actions at a discrete time t, respectively. The policyholder receives labor income y_t at t^-. Still at time t^-, withdrawal, consumption and term life policy purchase decisions are made. Then the insured determines the investment allocation decision between the fixed and the variable subaccounts at t^+, which is still at time t but after he obtains income, makes decisions regarding withdrawal, consumption and term life policy purchase. We also assume that the beneficiary receives the inheritance (value in the VA account and face amount of term life policy) immediately at t^+ just after the insured dies at t^+. We rewrite the Bellman equation in the Model section to a two-stage Bellman equations. From t^- to t^+ (i.e. the first stage), the insured consumes and gets the utility. If the insured dies, at t^+ his beneficiary will receive VA amount guaranteed by GMDB b_{t^+} and face amount of term life policy F_t^c. Therefore, the first stage equation is as follows,

$$(4) \quad V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t, P_t} \big\{ u(c_t) + \zeta(1-\phi_t) v_B(b_{t^+} + F_t^c) + V_{t^+}(a_{t^+}, b_{t^+}) \big\}.$$

Since the GMDB level will be reduced proportionally by the insured's withdrawal at t^- from the variable annuity account, we rewrite the above equation as follows,

$$(5) \quad V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t, P_t} \big\{ u(y_t + d_t - P_t) + \dots$$
$$\dots + \zeta(1-\phi) v_B(b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}} + F_t^c) + V_{t^+}(a_{t^-} - d_t, b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}}) \big\}.$$

In the new equation above, we use all asset level variables subscripted by t^-. With the CRRA assumption, we can numerically generate a constant factor ψ_t at period t to make $v_B(b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}}) = \psi_t(b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}})^{1-\gamma}$, given $\psi_t < 0$ if $\gamma > 1$, and $\psi_t > 0$ if $\gamma < 1$. And Equation (5) can be rewritten as

$$(6) \quad V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t, P_t} \big\{ u(y_t + d_t - P_t) + \dots$$
$$\dots + \zeta(1-\phi) \psi_t (b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}} + F_t^c)^{1-\gamma} + V_{t^+}(a_{t^-} - d_t, b_{t^-} \frac{a_{t^-} - d_t}{a_{t^-}}) \big\}.$$

Then, at the 2nd stage, i.e. from t^+ to $(t+1)^-$, the insured maximizes the value function $V_{t^+}(a_{t^+}, b_{t^+})$ from optimally allocating between the two subaccounts and $V_{t^+}(a_{t^+}, b_{t^+})$ is the expected discounted value of $EV_{t+1^-}(a_{t+1^-}, b_{t+1^-})$.

$$(7) \quad V_{t^+}(a_{t^+}, b_{t^+}) = \max_{\omega_t}\left\{\beta\phi_t EV_{t+1^-}(a_{t+1^-}, b_{t+1^-})\right\}.$$

Finally, the terminal value at time $(T+1)^-$ is as follows

$$(8) \quad V_{T+1^-}(a_{T+1^-}) = \sum_{t=T+1}^{T_{max}} \beta^{t-(T+1)}(\textstyle\prod_{i=T+1}^{t-1} \phi_i)u(\bar{c}).$$

3 Numerical Methodology

Since our model has no closed form solution, we will apply numerical analysis. We assume the policyholder makes decisions monthly, i.e. the policyholder deposits a lump sum amount into his VA account at the beginning of age 35, and he makes all decisions at the beginning of every month until his retirement age (at the beginning of age 65). Following Gao and Ulm (2015), we use the two-stage Bellman equations and apply a 2-Dimensional lattice. We solve the utility optimization problem by backward induction from age $t = T$ (at the beginning of age 65, $T = 360$) to $t = 1$ (at the beginning of age 35). We discretize periodic fund value $A = [0, a_{max}]$, into 51 nodes[4], and return of premium and roll-up benefit level $B = [0, b_{max}]$, into 51 nodes as well[5]. Therefore, at any given period t, we set a state space with 51 asset levels by 51 GMDB levels, and the entire state space is 51 asset levels by 51 GMDB levels by 360 months from age 35 to 65. All state variables are denoted as $(\cdot)_{t^-}$, $(\cdot)_{t^+}$. The backward methodology can be described as follows:

[4]Between 2 neighboring nodes, we define jump size following a trinomial tree setting, i.e. let $a_1 = (1/u)^{25}$, where $u = e^{\sigma\sqrt{3\Delta t}}$ is the jump size, and $a_i = a_1 \times u^{i-1}$, for $i = 2, 3, \cdots, 51$. If annual volatility $\sigma = 15\%$, then the monthly jump size $u \approx 1.07788$. The lowest asset level is $a_1 = 0.1534$, the middle node is $a_{26} = 1$, and the largest node is $a_{51} = 6.52$

[5]we assume the value of every node in $\hat{A}$ is equal to the value of every node in $\hat{B}$, i.e. $a_i = b_i$ for $i = 1, 2, \cdots, 51$.

From $(t+1)^-$ to t^+, we decide the insured's allocation between the fixed and variable subaccounts. Following Gao and Ulm (2012), we use a trinomial lattice with probabilities p_u, p_d, and p_m as follows,

$$(9) \quad p_u = \frac{A\omega^2 + B\omega + C}{(u-1)(u-d)},$$

$$(10) \quad p_d = \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d-1} - \frac{A\omega^2 + B\omega + C}{(d-1)(u-d)},$$

$$(11) \quad p_m = 1 - \frac{A\omega^2 + B\omega + C}{(u-1)(u-d)} - \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d-1} + \frac{A\omega^2 + B\omega + C}{(d-1)(u-d)},$$

where

$$d = \frac{1}{u},$$

$$A = e^{(2r+\sigma^2)h} - 2e^{(r+g)h} + e^{2gh},$$

$$B = (e^{rh} - e^{gh})(2e^{gh} - d - 1),$$

$$C = (e^{gh} - 1)(e^{gh} - d).$$

Given GMDB level b_i, let

$$(12) \quad a_{t+1^-} = a_{t^+} \times \begin{pmatrix} d & 1 & u \end{pmatrix}.$$

Then at any given GMDB level b_i, we derive

$$(13) \quad V_{t^+}(a_j, b_i) = \max_{\omega_t} \{ \beta\phi(p_u V_{t+1^-}(a_{j+1}, b_i) + p_m V_{t+1^-}(a_j, b_i) + p_d V_{t+1^-}(a_{j-1}, b_i)) \},$$

for $i = 1, 2, \cdots, 51$ and $j = 1, 2, \cdots, 51$, where $V_{t^+}(\cdot, \cdot)$ is a 51×51 matrix at time t^+. By taking the first derivative on ω, and the optimal ω is:

$$(14) \quad \omega^\star = -\frac{(d-1)BV_{t+1,j} - V_{t+1,j-1} + (u-1)(V_{t+1,j} - V_{t+1,j+1})[(u-d)(e^{rh} - e^{gh}) - B]}{2A[(d-1)(V_{t+1,j-1} - V_{t+1,j}) + (u-1)(V_{t+1,j} - V_{t+1,j+1})]}.$$

From t^+ to t^-, we then maximize V_{t^-} to determine the insured's optimal withdrawal d_t and term life premium P_t according to *Eq. (6)* by using cubic spline interpolation. We repeat the processes from $(t+1)^-$ to t^+ and t^+ to t^- respectively until age 35.

4 Numerical Results

In this section, we present the sensitivity test results by incorporating unemployment risk or health uncertainty. We assume the following parameter values as in Table 1.

Table 1: Common Parameters in the Base case

Strength of Bequest Motive	ζ	0.5
Subjective Discount Rate	β	0.97
Risk Free Rate	r_f	3%
Growth Rate of Fixed Subaccount	g	4%
Expected Return of Risky Asset	r	7%
Volatility of Risky Return	σ	15%
GMDB roll-up rate	r_p	0
Annual Mortality Rate	μ	1994 MGDB table [6]
Monthly Income	y	0.01

4.1 Unemployment Risk

Now the assumption of unemployment risk is added to our model. Therefore, there are two states of labor income: State 1. $y = 0.01$ per period (month), the policyholder is employed and receives a positive compensation $\tilde{y} > 0$; State 2. $y = 0$, the policyholder is unemployed and receives no income. These two states can be transited by following a Markov transition probability matrix as follows,

$$\Pi(y_{t+1}|y_t) = \begin{pmatrix} 0.95 & 0.05 \\ 0.3 & 0.7 \end{pmatrix}$$

If the policyholder is employed at time t, he has a 95% chance to remain employed and a

[6]Since the table provides annual mortality and we need to use monthly mortality rate, we assume uniform distribution of deaths over the year, i.e. any given month 1/12 of the people die.

5% chance to become unemployed at time $t+1$; if the policyholder is unemployed at time t, his probability of finding a job at $t+1$ is 30% and his probability of remaining unemployed at $t+1$ is 70%. The steady state probability of employment is $\pi_1 = \frac{6}{7}$ and the steady state probability of unemployment is $\pi_2 = \frac{1}{7}$. The above Markov transition probability matrix defines the base economy case. Since unemployment risks can vary, we test the policyholder's behavior under different economic situations. We define a good economy as one in which the employed insured is not likely to get unemployed, and a bad economy as one in which there are few job openings, i.e. an unemployed insured has difficulty finding a job. The Markov transition probability matrix of the good market case is as follows,

$$\Pi(y_{t+1}|y_t) = \begin{pmatrix} 0.995 & 0.005 \\ 0.3 & 0.7 \end{pmatrix}$$

In the good market case, if the policyholder is employed at time t, he has a 99.5% chance of remaining employed and a 0.5% chance of becoming unemployed at time $t+1$; if the policyholder is unemployed at time t, his probability of finding a job at $t+1$ is 30% and his probability of remaining unemployed at $t+1$ is 70%. The steady state probability of employment is $\pi_1 \approx 98.36\%$ and the steady state probability of unemployment is $\pi_2 \approx 1.64\%$.

The Markov transition probability matrix of the weak market case is as follows,

$$\Pi(y_{t+1}|y_t) = \begin{pmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{pmatrix}$$

Under the weak economy scenario, if the policyholder is unemployed at time t, he has a 90% chance of staying unemployed and a 10% chance of finding a job position at time $t+1$. The steady state probability of employment is $\pi_1 \approx 66.67\%$ and the steady state probability of unemployment is $\pi_2 \approx 33.33\%$.

4.1.1 Allocation and Withdrawal Choices

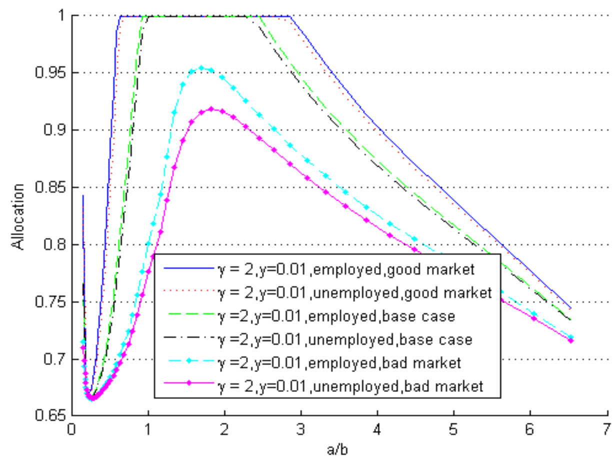

Figure 1: Age 45 allocation with $\gamma = 2$

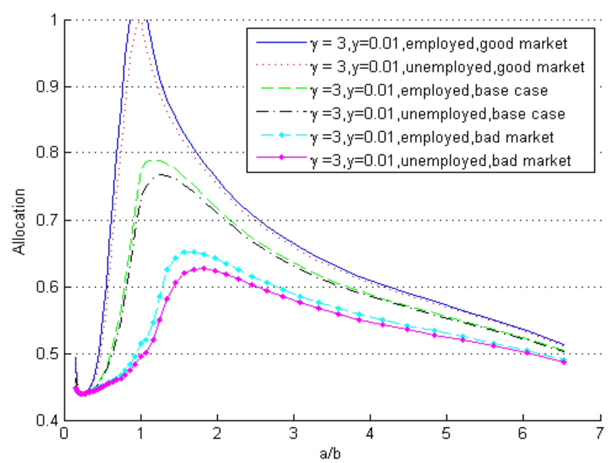

Figure 2: Age 45 allocation with $\gamma = 3$

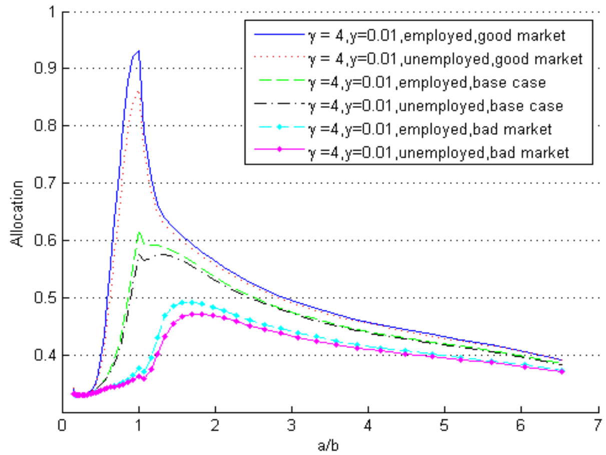

Figure 3: Age 45 allocation with $\gamma = 4$

Figures 1-3 (age 45 allocation with different economy scenarios) detail the optimal allocation choices for the insured given unemployment risk during a particular year (age 45). Since the insured is rational, he will take the unemployment risk into account when he allocates his wealth to maximize his utility. Acknowledgment of the unemployment risk by the insured ensures that he has a lower expectation for his human capital than if he were in a fixed income situation. One can observe that the economic environment significantly influences the policyholder's allocation choices. If the economy is good, the policyholder expects more human capital and is willing to take more financial risk and allocate more money to the risky subaccount. If the economy is weak, the policyholder who worries about unemployment will be more averse to financial risk and allocate less money to the risky subaccount compared to both the good and base scenarios.

A further observation is that there is not a large difference between allocations made by employed or unemployed policyholders in the good and base scenarios while the difference becomes greater in the weak economy case. Given the Markov transition probabilities above, the unemployed policyholder is expected to wait only 3 months (1/0.3) for a new job for both the good and base cases. Since three months is a short period, therefore, a rational policyholder will not reduce the risky allocation by much. However, the unemployed policyholder is expected to wait as long as 10 months (1/0.1) for a new job in the weak case. It therefore makes sense for him to reduce the risky allocation to keep liquidity/safe money for current consumption.

The hump shape around the "at-the-money" area is caused by two effects: The first is the GMDB effect. The insured allocates his asset according to the Merton (1969) allocation. However, due to the existence of the GMDB option, the beneficiary may prefer a more agressive allocation strategy around the "stock-at-strike" level in our study, because the downside risk of his benefit is protected by the GMDB. When the GMDB is far out-of-the-money, the GMDB protection is not valuable and the allocation is close to the Merton allocation. When the GMDB is far in-the-money, the beneficiary is indifferent to the allocation preference because she will receive the value at the strike level. Thus, the insured also prefers the Merton allocation. In conclusion, the beneficiary has a strong preference to have an agressive allocation near the at-the-money area. We call this an "argument" between the insured and his beneficiary.

The second is the human capital effect. Individuals' asset allocation varies with their life-cycle and their income according to traditional financial planning advice. We assume employment income (human capital), which is a safer capital than financial assets, thus the insured would like to invest more in risky financial assets. The hump shape around the at-the-money area can be partially explained by the human capital effect. When the asset level is very large, the percentage of asset value represented by human capital is reduced and the allocation percentage to the risky subaccount is reduced correspondingly. However, the figures 1-3 are counter-intuitive when the asset level is low (as the asset value goes below the GMDB strike level). The risky asset allocation should be high as the percentage of riskless assets (human capital) is great, but our figures show a small allocation percentage. This result occurs because our model assumes the policyholder consumes all periodic income during the current period. The "rational" policyholder realizes he is not able to save this part of wealth for his retirement or transfer it to his beneficiary but over-consume it. He will become conservative in asset allocation. As the asset level is too low, the insured will prefer to take risk and increase his allocation in the risky subaccount because he has little to lose but much to gain.

In addition to economic conditions, the policyholder's risk aversion levels also matter for his allocation decision. As the risk aversion increases, the insured will invest less in the variable subaccount. Given the value of the coefficient of relative risk aversion ($\gamma = 2$), the risky allocation in Figure 1 is the greatest for all 3 economic scenarios compared to Figure 2 ($\gamma = 3$) and 3 ($\gamma = 4$).

Figures 4-6 show the effect of changing economic scenarios on the policyholder's withdrawal decisions from the VA account. When the policyholder is unemployed, we assume all of his living expenses will be from the withdrawal of the VA account value. Therefore, we observe that the proportion of withdrawals in unemployed states are much greater than in employed states. In addition, the more money the unemployed policyholder has, the smaller percentage he withdraws. When the asset level is low, the insured has to withdraw a large proportion to cover his living expenses. As the asset level increases, the absolute value of consumption is still increasing, but the consumption ratio is decreasing. When the policyholder is employed, he is reluctant to withdraw from the VA account when the GMDB is in-the-money, because if he withdraws, the GMDB strike level will be reduced proportionally and this will hurt his beneficiary. As the asset level increases near or above the GMDB strike level, he starts to withdraw. In addition, we find the policyholder prefers to consume more in both employed and unemployed states in good economic conditions, because when the economy is good, the insured expects he will have more human capital and therefore consume more to improve his living standards.

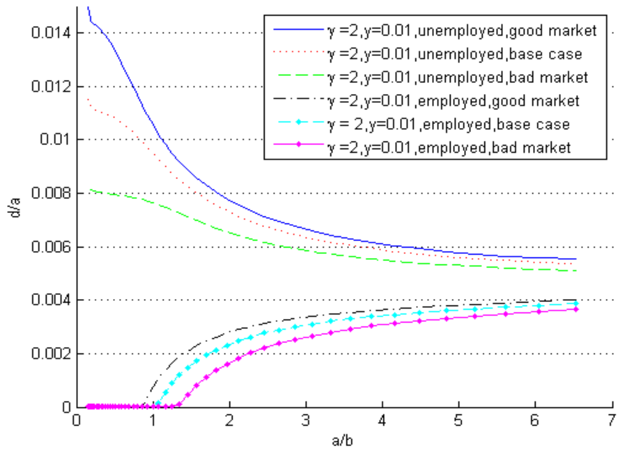

Figure 4: Age 45 allocation with $\gamma = 4$

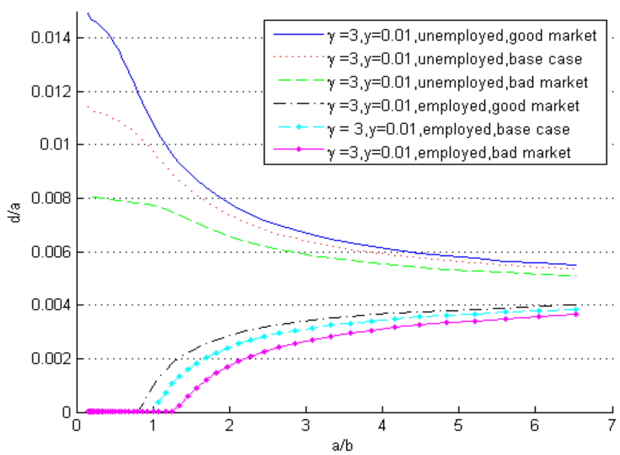

Figure 5: Age 45 allocation with $\gamma = 4$

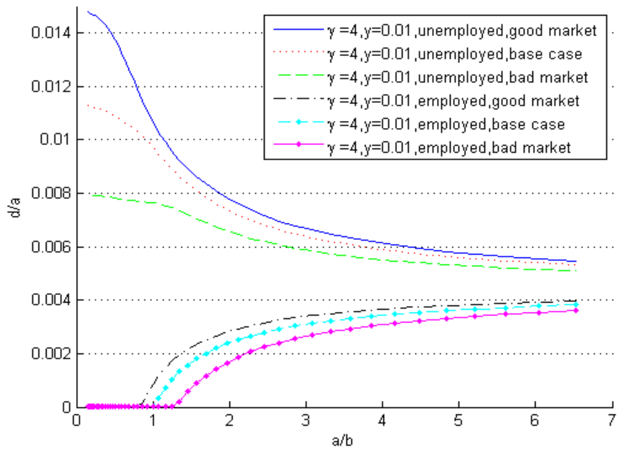

Figure 6: Age 45 allocation with $\gamma = 4$

Figures 7-9 plot the at-the-money allocation in different economic scenarios through the entire accumulation period (from age 35 to age 65). One can observe a downward trend in risky assets allocation in all economic scenarios. As the policyholder ages, he will realize his probability to die before retirement will reduce gradually and he will be more likely to consume the VA assets. As a result, he will decrease the allocation to risky assets, and the allocation will converge to the Merton allocation at age 65. In addition, one can observe some obvious results. First, good market conditions encourage the policyholder to take more risk. Second, less risk averse policyholders allocate more money to the risky subaccount. Finally, the policyholder is willing to take more risk in employed states than in unemployed states.

Figures 10-12 show at-the-money withdrawal proportion from age 35 to age 65. Since all consumption comes from withdrawing from the VA account in the unemployed states, one can observe a higher withdrawal ratio than in the employed states. In addition, as explained in Figures 4-6, the policyholder withdraws most in the good market scenario and least in the weak market scenario.

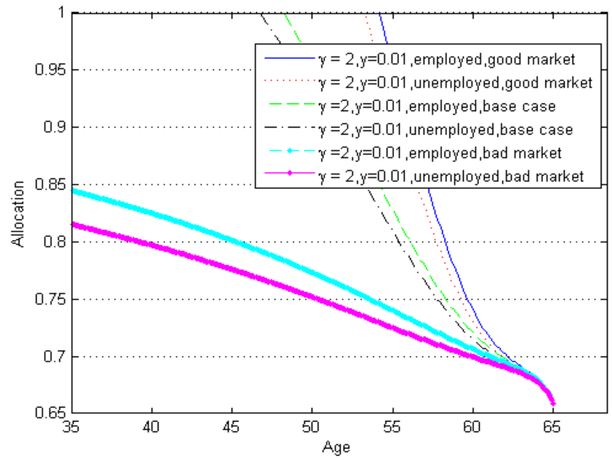

Figure 7: At-the-money allocation with $\gamma = 2$

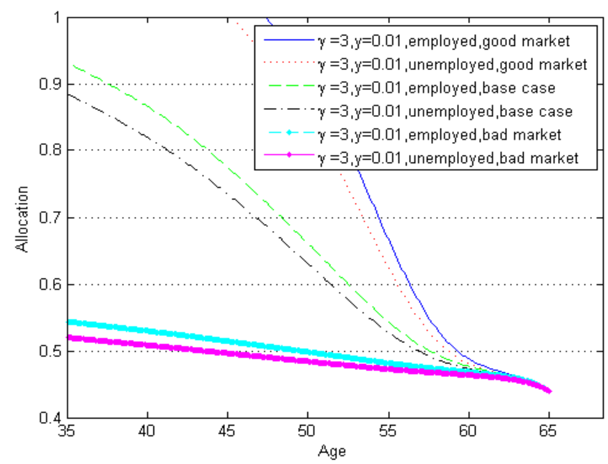

Figure 8: At-the-money allocation with $\gamma = 3$

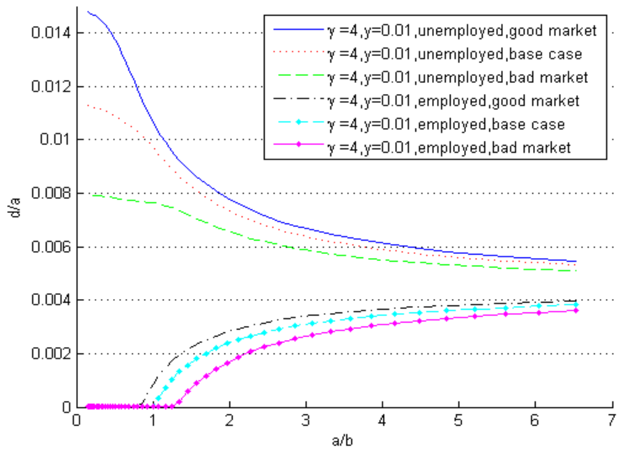

Figure 9: At-the-money allocation with $\gamma = 4$

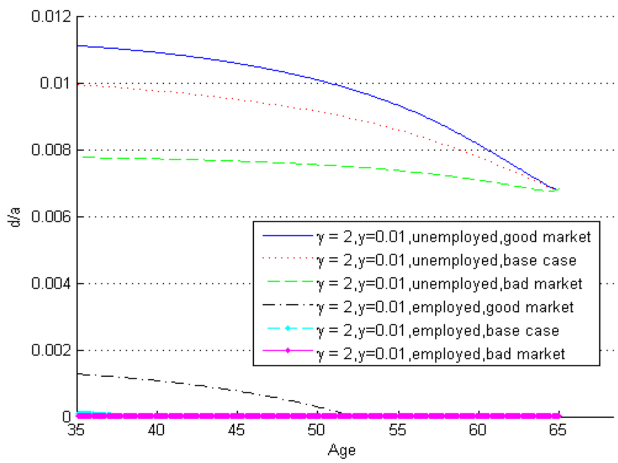

Figure 10: At-the-money withdrawal with $\gamma = 2$

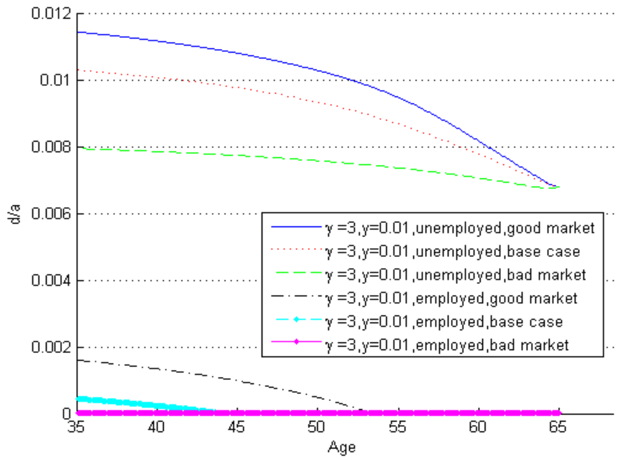

Figure 11: At-the-money withdrawal with $\gamma = 3$

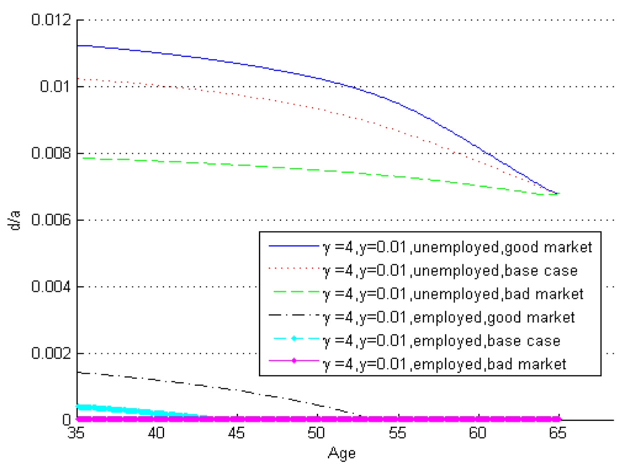

Figure 12: At-the-money withdrawal with $\gamma = 4$

4.1.2 Term Life Insurance Demand

In this section, We examine the policyholder's term life demand. We assume all policyholder's adjustments are made monthly. Figures 13 and 14 plot term life demand in the base market conditions. There are several observations: First, the value of the loading factor η influences the amount of the term life purchases. As term life policies become expensive (η increases from 0 to 0.2), term life demand decreases. In addition, one can observe that the demand decrease is smaller at low (less than 1) asset levels than high (greater than 1) asset levels. The demand decreases dramatically as asset levels becomes very high because wealthy people have alternative choices for leaving bequests. Thus, as term life becomes expensive, wealthy people will choose other wealth transfer channels. However, poor people have less investment vehicle choices. Although η increases, term life policies may still be cheap and efficient wealth transfer vehicles for poor people. Second, the more risk averse the policyholder is, the more term life he would like to purchase. When the risk aversion level at $\gamma = 4$, the term life policy demand is the greatest; while when $\gamma = 2$, the term life policy demand is the lowest. Third, employment states matter in the term life purchase decision, especially when the asset level is low. People purchase term life policies to protect against the loss of human capital. Thus, if the policyholder is unemployed, he will have a lower demand for term life policies. Especially, when the asset level is very low, the policyholder may regard other necessity consumption as the first priority and reduce his term life purchases accordingly. Finally, unemployment risk influences the amount of term life demand. Figures 15 and 16 plot term life demand when the economic condition is good, and Figures 17 and 18 plot term life demand when the economic condition is weak. In the good economic scenario (Figures 15 and 16), one observes the strongest term life demand; while in the weak economic scenario (Figures 17 and 18), one observes the weakest term life demand.

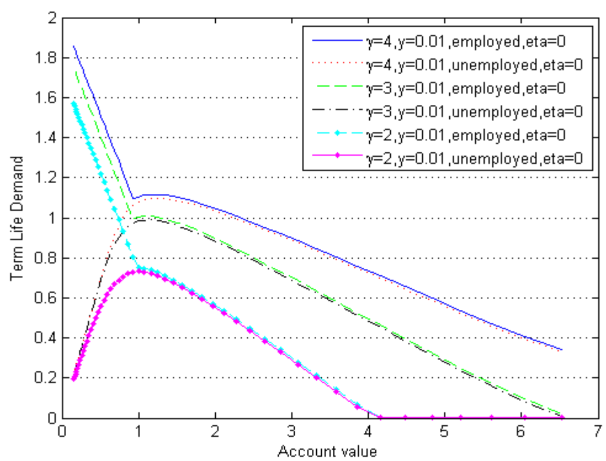

Figure 13: Base case term life demand with $\eta = 0$

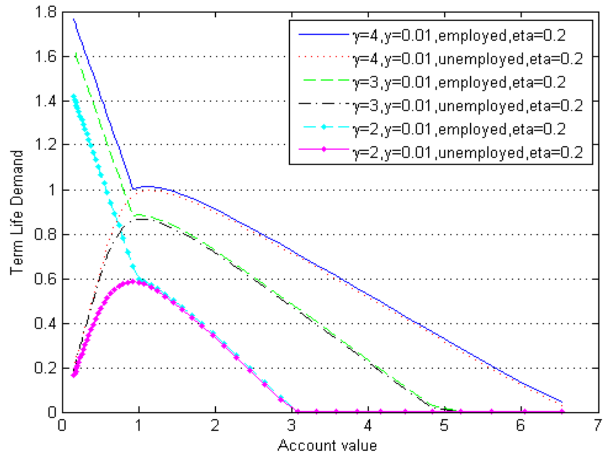

Figure 14: Base case term life demand with $\eta = 0.2$

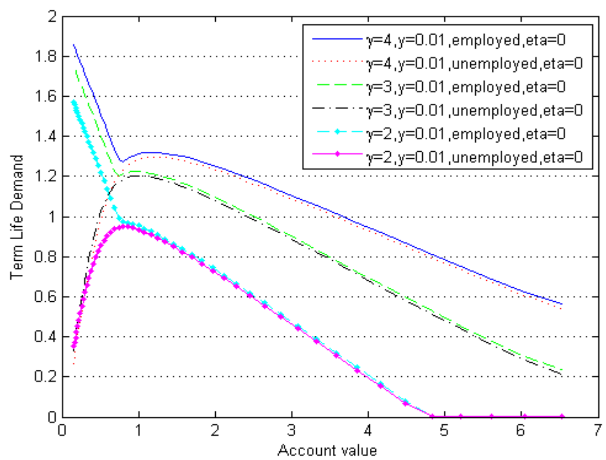

Figure 15: Good market term life demand with $\eta = 0$

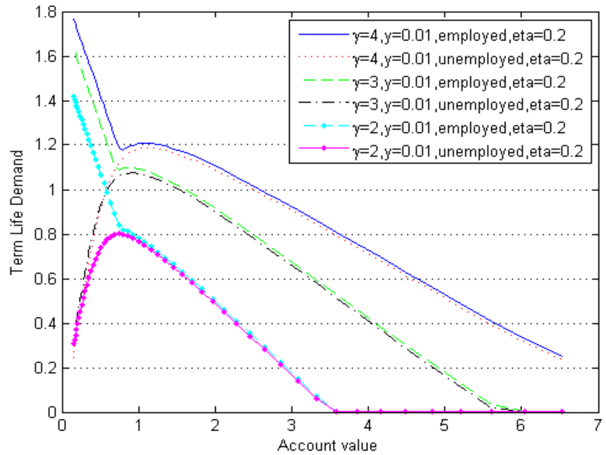

Figure 16: Good market term life demand with $\eta = 0.2$

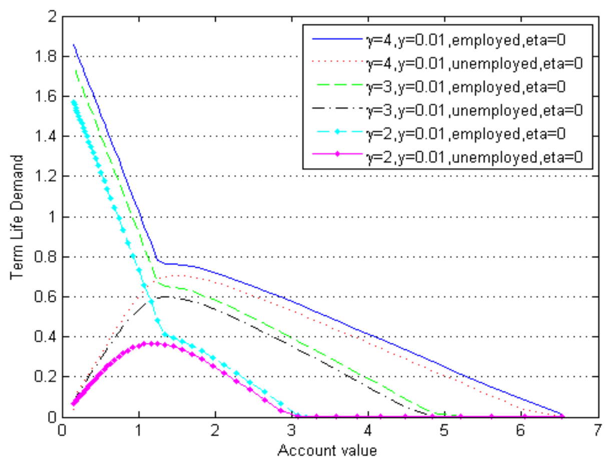

Figure 17: Weak market term life demand with $\eta = 0$

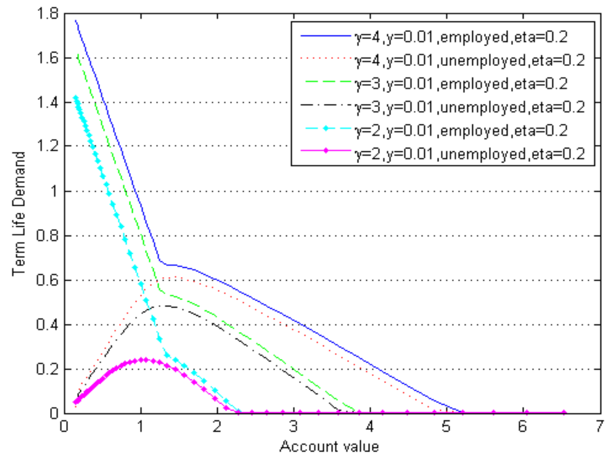

Figure 18: Weak market term life demand with $\eta = 0.2$

4.2 Health Uncertainty and Term life Insurance Demand

We now add health uncertainty to the model. In this case, the policyholder may get sick, which has two effects. First, his working ability will be effected and he will earn less while sick. Second, his mortality risk will increase and term life life purchases will become more expensive. We set two health states (H) for the policyholder: healthy and unhealthy. These two states can be transited by following a Markov transition probability matrix as follows,

$$\Pi(H_{t+1}|H_t) = \begin{pmatrix} 0.95 & 0.05 \\ 0.3 & 0.7 \end{pmatrix}$$

If the policyholder is healthy at this period, he has a 95% chance of remaining healthy and a 5% chance of being unhealthy in the following period; if the policyholder is unhealthy in this period, his probability of recovering from sickness at next period is 30% and his probability of remaining in the unhealthy state in the next period is 70%. In these two health states, the mortality risks vary. If the policyholder is sick, the mortality risk will be greater than in the healthy state. We focus on term life insurance demand given the existence of health uncertainty in this section[7].

[7]To avoid replication, it may be more worthy to focus on the discussion of term life insurance demand.

In Figures 19 and 20 [8], we plot the term life policy demand with health uncertainty. If the policyholder is unhealthy in the current period, his mortality risk in the current period will be doubled/tripled and his periodic income will be discounted by 50%. Since the increased mortality risk results in lower working ability and greater term life premium in the unhealthy state, it reduces the insured's demand for term life policies. Especially when the unhealthy insured has less savings, one can observe a large drop in the term life demand. However, when the account value is very low, the unhealthy insured will increase the amount of term life purchases because he has a bequest motive and a term life policy may be the best way to leave a bequest in such a situation. When the insured is rich, he is not as sensitive to the cost of term life as when he is poor. In addition, consistent with the previous analysis, the term life demand will decrease as the account value increases.

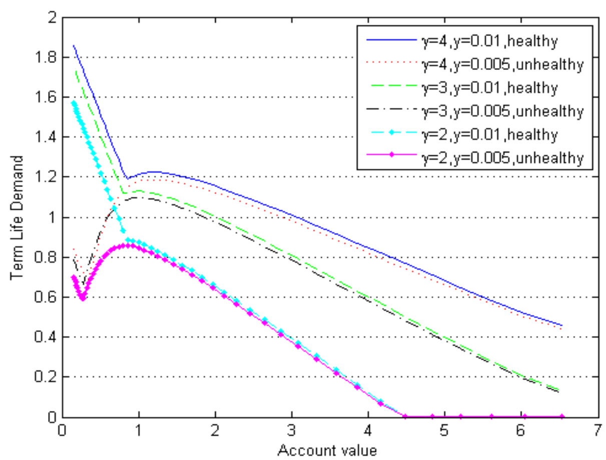

Figure 19: Term life demand with doubled mortality at the sick state

[8] We use $\eta = 0$ in the analysis of Figures 19 and 20.

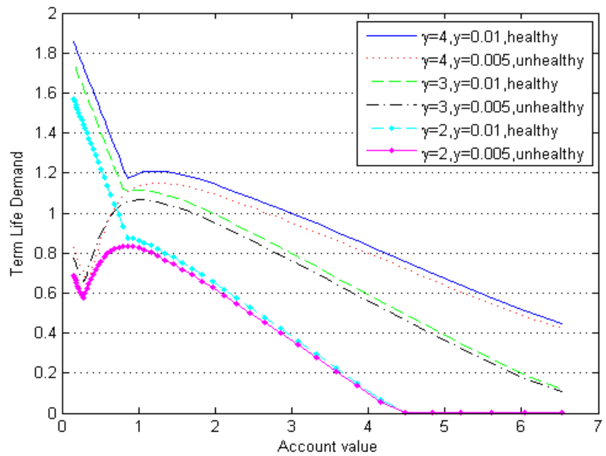

Figure 20: Term life demand with tripled mortality at the sick state

4.3 GMDB Prices

Following Gao and Ulm (2012), we derive the GMDB "fair" prices by assuming the insurer applies the strategy of assuming the insured makes optimal choices. In addition, we follow Gao and Ulm (2015) to derive GMDB prices which the insured is willing to pay.

Table 2 compares the at-the-money GMDB fair price (basis points) to the GMDB price which insured is willing to pay at age 35. In the case of unemployment risk, one can observe the following results. First, as the risk aversion level increases, the GMDB fair price generally decreases. This occurs because as the risk aversion level increases, the risky allocation decreases and therefore the account value volatility decreases and the GMDB option price decreases. However, if the policyholder is more risk averse, he may have more demand for the GMDB option and therefore be willing to pay a higher price for GMDB protection. Second, the economy matters in GMDB pricing and the policyholder's decision to purchase. If the economy is good, the policyholder will allocate more to the risky subaccount and the insurer will charge a correspondingly higher price for the GMDB option. On the other hand, the policyholder is willing to pay less for GMDB protection as the economy improves, because they care less about equity risk during the prosperous period. Third, the term life loading factor influences the policyholder's willingness to pay for the GMDB option. Both the GMDB option and a term life insurance policy can be regarded as a pair of substitute goods, because both products can help a policyholder hedge the risk of the loss of human capital. Therefore, if the term life policy becomes expensive due to higher loading, people will be willing to pay more to obtain the GMDB option instead. However, we can also find the fair price charged by insurer does not change much, because the insurer's charge depends heavily on policyholder's allocation and lapse decisions which do not change much. Forth, unemployed policyholder is willing to pay a higher price for the GMDB option in any underlying economic situation. Because he has far less human capital and has to rely on the savings in the VA account. He has more incentive to hedge the downside risk with the GMDB. Finally, because of the combined effects of these factors, one can observe that the insured's willingness to pay exceeds the insurer's offer price in some scenarios (shown in bold in Table 2).

In the case of health uncertainty, we find the following: First, as in the unemployment risk case, the GMDB premium charged by insurer is negatively correlated with risk aversion level while the policyholder's willingness to pay is positively correlated with risk aversion level. Second, if the mortality risk increases and makes the term life policy expensive, the GMDB will be more valuable to the insured and he is willing to pay more for it. Third, there is only a very slight difference in the willingness to pay between healthy and unhealthy insureds. Finally, the risk aversion level plays an important role in filling the gap between the price the insurer would like to charge and the price assessed from the insured's perspective.

Table 2: GMDB fair price vs. insured's willingness to pay at age 35

γ	Market Condition	η	fair price	willingness to pay
Unemployment Risk				employed/unemployed
4	good market	0	3.3520	1.2211/1.4241
4	good market	0.2	3.3633	1.4311/1.5872
4	base case	0	**1.2733**	**1.4545/1.6044**
4	base case	0.2	**1.2779**	**1.7741/1.8977**
4	weak market	0	**0.1834**	**2.4308/2.8751**
4	weak market	0.2	**0.1856**	**2.5528/3.5476**
3	good market	0	4.9274	1.0887/1.1505
3	good market	0.2	4.9335	1.2075/1.3876
3	base case	0	2.3978	1.3733/1.4727
3	base case	0.2	2.4070	1.6636/1.8119
3	weak market	0	**0.5373**	**2.3602/2.7206**
3	weak market	0.2	**0.5416**	**2.5125/3.1932**
2	good market	0	6.5081	0.8667/1.0023
2	good market	0.2	6.5130	1.0000/1.1333
2	base case	0	4.8513	1.3333/1.4444
2	base case	0.2	4.8587	1.5630/1.7283
2	weak market	0	2.2786	2.2273/**2.5447**
2	weak market	0.2	2.2862	**2.4545/2.8318**
γ	Mortality	Income	fair price	willingness to pay
Health Uncertainty				(healthy/unhealthy)
4	double	half	2.7182	2.2537/2.2310
4	triple	half	2.7290	2.3337/2.3180
3	double	half	4.2390	1.5245/1.4840
3	triple	half	4.2594	1.6135/1.5840
2	double	half	5.9678	1.0267/0.9911
2	triple	half	6.0216	1.0459/1.0271

5 Conclusions

In the paper we examine the effect of labor income and health uncertainty on the optimal choices of policyholders with Guaranteed Minimum Death Benefits. We find that being unemployed has a significant effect on a policyholders demand for term life insurance. In particular, unemployed individuals with low account values are unwilling to sacrifice current consumption for beneficiary protection. This, in turn, has an effect on a policyholder's willingness to pay for the GMDB protection which is now very valuable to unemployed policyholders with low account values. In contrast to previous studies, we find that this effect causes very risk-averse policyholders in weak labor markets to be willing to pay the GMDB fees to receive the protection.

In this study, we do not examine the interaction between weak job markets and poor stock market performance. We would expect low accounts values and unemployment to be correlated in the real economy and this would only serve to increase the range of overlap between an insured's willingness to pay and the fair price of the GMDB. This would be a promising avenue for further research.

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