The Marginal Cost of Risk and Capital Allocation in a Property and Casualty Insurance Company

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In this paper, we introduce a multi-period profit maximization model for a property and casualty (P&C) insurance company, and use it for determining the marginal cost of risk and resulting economic capital allocations. In contrast to previous literature and as an important innovation, our model features a loss structure that matches the characteristics of a P&C company, comprising short-tailed and long-tailed business lines. In particular, we take into account the loss history and loss development years. As an example application, we implement the model using two P&C insurance business lines and two development years, and using NAIC loss data for calibration. Our numerical results demonstrate how loss history affects the marginal cost and capital allocations.

*Keywords:* loss triangle; chain ladder; profit maximization; marginal cost of risk; capital allocation; high dimensional dynamic programming.
1 Introduction

Allocating capital to business line is one of the key actuarial tasks in an insurance enterprise. More precisely, since the risk associated with different business lines has to be supported by capital and since capital is costly, this process is a key ingredient to setting prices and measuring the performance of different lines. Thus, it is no surprise that there exists a large number of related papers in the actuarial and broader insurance literatures, although the focus is primarily on technical aspects of how to divide risk capital given by an arbitrary risk measure (see e.g. Albrecht, 2004; Bauer and Zanjani, 2013; McNeil et al., 2015, Section 8.5 and references therein). In a sequence of papers, Bauer and Zanjani (2015a,b) argue that capital allocation should flow from the company’s economic objectives. In particular, they show that economic capital allocations can be derived from the marginal cost of risk within the organization’s profit maximization problem. However, while Bauer and Zanjani provide a suitable conceptual framework for allocating capital for an insurance company, their models do not fit the basic characteristics of a property and casualty (P&C) insurer – where capital allocation is most relevant. Providing and exploring such a model framework are the key contributions of this paper.

One of the key characteristics is the loss structure of a P&C insurer. More precisely, in many of the business lines, claims can take several years to get reported and paid. Indeed, this may take close to a decade within so-called long-tailed lines such as workers compensation insurance. The prevalence of “loss history” leads to particular concerns for P&C insurers: in addition to the “shock” from exposure to risk in the current year, they also receive “after shocks” from exposure to risk taken in the previous years. This impacts the dependence structure of the loss payments through time as well as the relevant “state space” for the insurer. As a consequence, the loss development affects optimal
insurance portfolio and capital allocation result. To explore these aspects, we integrate a general P&C loss structure given via so-called *loss triangles* with a multi-period profit maximization model for an insurer that economizes on different capitalization options: preserving internal capital, raising external capital, and raising emergency capital (Bauer and Zanjani, 2015b).

We solve our model numerically in a two lines and two development years specification. More precisely, we consider a business with a commercial automobile insurance line and a workers’ compensation line, where we assume that the losses develop according to a Chain-Ladder model with jointly normal innovations Mack (1993). We calibrate the model using NAIC data and we implement the firm’s profit maximization problem by dynamic programming on a discretized state space.

In line with Bauer and Zanjani (2015b), we find that the value of a P&C insurer is concave with an optimal point that results from balancing profit expectations and capital costs. However, in our setting, we also find that the loss history has a considerable affect on the value function. In particular, a high past exposure and a high past loss realization for the longer-tailed workers’ compensation line will have a larger effect relative to the shorter-tailed commercial auto line. Generally, the consideration of loss history is important in our example calculations and alternative methods lead to inefficient outcomes.

This paper relates to several strands of literature. First, we bring together economic approaches for capital allocation in financial institutions with actuarial loss forecasting methods, for the purpose of deriving economic allocations in P&C insurance. A key contribution with regards to the former literature is Froot and Stein (1998), who first propose a model of capital allocation with economic foundation. Later literatures including (Zanjani, 2002, 2010) extend the economic capital allocation to insurance. The paper also

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1The Chain-Ladder method is one of the predominant methods for loss reserving in non-life insurance. We refer to the textbooks by Wuethrich and Merz (2008) and Taylor (2010) for details.
relates to some of the extensive actuarial literatures on claims reserving methods (Taylor and Ashe, 1983; Renshaw, 1989; Mack, 1993; Wüthrich and Merz, 2008; Taylor, 2012). The numerical techniques required to solve the multi-period model relate to literatures in high-dimensional state space dynamic programming and numerical techniques literatures (Tanskanen and Lukkarinen, 2003; Ljungqvist and Sargent, 2012).

This paper features the following sections: Section 2 presents a general model of multi-period profit maximization with a general loss structure in a P&C company; Section 3 presents an implementation of the model with two lines and two development years, marginal costs of risk calculation and results; Section 4 concludes our findings.

2 A General Multi-Period Profit Maximization Model with History and Development Years

2.1 Loss Structure for a P&C Company

Setting up a profit maximization framework for a P&C company requires modeling the asset and the liability side. For now, we assume the company’s assets bear no risk. We assume that all the uncertainty is captured by the liability side, modeled via the claim payment amounts in this paper. Hence, it is essential to capture an appropriate loss structure, which differs from that of other companies.

A P&C company writes and sells new insurance contracts in each of its business lines every year, called accident year, during which accidents occur and losses are reported. However, some of the losses are not reported until the next year or even years after. Furthermore, only some portions of payments are settled in the accident year. Generally the rest of the unrealized payments will take several years to settle. The lags in reporting and paying losses are accounted by development years. This loss structure is typically represented via a so-called loss triangle, with one triangle recording incurred (reported)
loss and one triangle recording paid loss. For example, to describe the paid loss, we consider a P&C insurance company with $N$ business lines with corresponding loss random variable $X_{i,j}^{(n)}$, with line identifier $n = 1, 2, ..., N$, accident year (AY) $t = 1, 2, ..., T$, and development year (DY) $j = 1, 2...d_n$. See figure 1 for an illustration:

<table>
<thead>
<tr>
<th>AY</th>
<th>DY</th>
<th>1</th>
<th>2</th>
<th>......</th>
<th>$d_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$X_{11}^{(n)}$</td>
<td>$X_{12}^{(n)}$</td>
<td>...</td>
<td>$X_{1,d_n}^{(n)}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$X_{21}^{(n)}$</td>
<td>...</td>
<td>$X_{2,d_n-1}^{(n)}$</td>
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<td>...</td>
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<tr>
<td>T</td>
<td>$X_{T1}^{(n)}$</td>
<td></td>
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</tbody>
</table>

Figure 1: Loss Triangle for A P&C insurer

In the paid loss triangle, $X_{12}^{(n)}$ to $X_{1,d_n}^{(n)}$ reflects amount paid (if positive, or amount received if negative) in addition to the initial payment in the year 1. In the paid loss triangle, there are payments to the losses incurred in the current year, as well as to the losses developed from the previous years. Specifically, payments in the same calendar year consist of the diagonal entries in the paid loss triangle. For example, payments in the year $T$ correspond to $(X_{1,d_n}^{(n)}, X_{2,d_n-1}^{(n)}, ..., X_{T1}^{(n)})$.

A incurred loss triangle can also be constructed as the table above. The $X_{i1}^{(n)}$ represents incurred losses in accident year $i$. Additional losses $X_{i2}^{(n)}, ..., X_{i,d_n}^{(n)}$ are reported as either positive or negative adjustments, to which the insurer is liable in the development year 2 to $d_n$. The initial losses and subsequent adjustments describe a loss development for the insurance contracts sold in the accident year $i$. From another perspective, a diagonal in the triangle entails loss reported in one calendar year. For example, in calendar year $T$, the insurer is exposed to the “shock” $X_{T1}^{(n)}$, as well as “after shocks” $(X_{1,d_n}^{(n)}, ..., X_{T-1,2}^{(n)})$, developed from the previous accident years.

To account for the loss development in each accident year, it is a common practice to
assume that cumulative paid loss has a Markov structure as the following:

\[
\mathbb{P}(M_{t,j}^{(n)}(X_{t_1}^{(n)}, \ldots, X_{t_j}^{(n)})) = \mathbb{P}(M_{t,j-1}^{(n)}(X_{t_1}^{(n)}, \ldots, X_{t,j-1}^{(n)}))
\]

The function \(M_{t,j}^{(n)}(X_{t_1}^{(n)}, \ldots, X_{t_j}^{(n)})\) can take various forms and range from one-dimensional to multidimensional. Indeed as it is possible to set \(M_{t,j}^{(n)}(X_{t_1}^{(n)}, \ldots, X_{t_j}^{(n)}) = (X_{t_1}^{(n)}, \ldots, X_{t_j}^{(n)})\), this assumption is without loss of generality. To ease the notations, from here on we will refer \(M_{t,j}^{(n)} = M_{t,j}^{(n)}(X_{t_1}^{(n)}, \ldots, X_{t_j}^{(n)})\), and a set of \(M_{t,j}^{(n)}\), \(\forall t = 0, 1, \ldots, T\) as \(M_{T}^{(n)}\).

Also, as it is a common practice, we assume independence across accident years. Markov and independence assumptions together fit most of the loss models in the P&C industry. It is possible to relax the independence assumption to allow cross-sectional correlations between accident years, with the cost of more complex derivations.

### 2.2 A General Multi-Period Profit Maximization Model

To fully describe the liability that a P&C company faces, we assume the following underwriting process. At the beginning of every period \(t\), the insurer chooses to underwrite certain portions in each line of business and charges premium \(p_t^{(n)}\) in return. The underwriting decision corresponds to choosing an exposure parameter \(q_t^{(n)}\) from a compact choice set \(\Phi_t^{(n)}\). For accident year \(i\), the company is going to pay \(q_i C_{t,d_n}\) in total, where \(C_{t,d_n}\) is the ultimate cumulative loss reported. This lump sum will be realized over the development years, but the payments are always dependent on the exposure parameter and paid loss random variables. Also note that in one period, the company’s indemnity entails exposure of losses paid in the current year, as well as losses developed from the past years and to be paid in the current calendar year. Thus for business line \(n\) in period
At the beginning of each period $t$, the company collects the full premium $p_t(n)$ that it writes on each line. Denote aggregate period premium by $P_t = \sum_{n=1}^{N} p_t(n)$. The company can raise capital $B_t \geq 0$ (or shed capital $B_t < 0$). The cost of raising capital $B_t$ is $c_1(B_t)$ if $B_t \geq 0$. Otherwise, there is no cost of shedding capital, or $c_1(B_t) = 0$ if $B_t < 0$. The company carries over capital $a_{t-1}(1 - \tau)$ from the last period, with $\tau$ denoting the unit frictional cost of capital. Thus, the company’s assets at the beginning of the period $t$ are $a_{t-1}(1 - \tau) + B_t - c_1(B_t) + P_t$. During period $t$, the assets are invested at a fixed interest rate $r$, compounded continuously. At the end of the period $t$, the company can decide whether to raise emergency capital $B^e_t \geq 0$ at a higher cost $c_2(.)$, with $c_2(x) > c_1(x)$, $\forall x \geq 0$, should the asset falls short of the aggregate indemnity $I_t$. The surplus of assets over aggregate indemnity, denoted by $a_t$, can be carried over to the period $t + 1$. Thus, we have the following law of motion for the company’s capital:

$$a_t = [a_{t-1}(1 - \tau) + B_t - c_1(B_t) + P_t] e^r + B^e_t - c_2(B^e_t) - I_t. \quad (1)$$

Additionally, the company cannot shed capital more than that it has available. For $a_{t-1} \geq 0$. We require that

$$B_t \geq -a_{t-1}(1 - \tau). \quad (2)$$

The objective function for each period can be derived using the revenue (premium collected), minus the costs (indemnity, frictional costs on carrying capital, and financing
costs). For each period, the expected aggregate indemnity is the following form:

\[ e_t = \mathbb{E}_{t-1} \left[ I_t I_{\{a_1 \geq 0 \ldots a_t \geq 0\}} + (a_t + I_t) I_{\{a_1 \geq 0 \ldots a_t < 0\}} \right]. \]

Note that if the company declares bankruptcy at the end of period \( t \), it pays out all its remaining assets \( a_t + I_t < I_t \).

Hence, the company’s period objective function \( f \) is:

\[
\begin{align*}
f(s_t = \{a_{t-1}, Q_{t-1}^{(1)}, \ldots, Q_{t-1}^{(N)}, M_{t-1,j-1}^{(1)}, \ldots, M_{t-1,j-1}^{(N)}\}, c_t = \{p_{t}^{(1)}, \ldots, p_{t}^{(N)}, B_t, B_{t}^{e}, q_{t}^{(1)}, \ldots, q_{t}^{(N)}\}) = & \\
\mathbb{E}\{e^{r}P_t - c_t - e^{r}(\tau a_{t-1} + c_1(B_t)) - c_2(B_{t}^{e})\} & \\
\mathbb{E}\left\{I_{\{a_1 \geq 0 \ldots a_t \geq 0\}}\{e^{r}P_t - I_t - e^{r}(\tau a_{t-1} + c_1(B_t)) - c_2(B_{t}^{e})\}ight. & \\
\left. - I_{\{a_1 \geq 0 \ldots a_t < 0\}}\{e^{r}(a_{t-1} + B_t) + B_{t}^{e}\}\right\}
\end{align*}
\] (3)

The state \( "s_t" \) contains all the variables that determine the state of the company at the beginning of period \( t \). The control \( "c_t" \) contains all the variables that the company choose for objective function maximization, also at the beginning of period \( t \). The insurance company’s ultimate objective is to maximize the future discounted cashflows, which corresponds to an infinite horizon problem of the following:

\[
\max_c \sum_{t=1}^{\infty} e^{-rt}f(s_t, c_t),
\] (4)

subject to (1), (2), and a generic premium function for each line \( n \):

\[
p_l^{(n)} = p_n(s_t, c_t).
\] (5)

According to Bertsekas (1995), optimization problem (4) is exactly resulting in the following Bellman equation. We can drop the time subscript and denote the next period state variables with a “prime” (standard notation in dynamic programming context).
Proposition 2.1 (Bellman Equation)

\[ V(a, Q^{(1)}, ..., Q^{(N)}, M^{(1)}_{j-1}, ..., M^{(N)}_{j-1}) = \max_{\{q^{(i)}\}, \{p^{(i)}\}, B, B^e} \E \left\{ 1_{\{a' \geq 0\}} \left[ P - e^{-r} I - \tau a - c_1(B) - e^{-r} c_2(B^e) \right. \right. \]
\[ + e^{-r} V(a', Q'(1), ..., Q'(N), M'_{j-1}, ..., M'_{j-1}) \left. \right] - 1_{\{a' < 0\}} (a + B + e^{-r} B^e) \right\} \]

Similar as in Bauer and Zanjani (2015b), we have three situations regarding the need of raising emergency capital $B^e$. A detailed proof is provided in the Appendix A.

Proposition 2.2

Assuming a linear cost $c_2(x) = \xi x$, $x > 0$, we have the following three states of the company regarding of raising emergency capital $B^e$.

Safe:

\[(a(1 - \tau) + B - c_1(B) + P)e^r \geq I \Rightarrow B^e = 0.\]

Save:

\[(a(1 - \tau) + B - c_1(B) + P)e^r < I \leq (a(1 - \tau) + B - c_1(B) + P)e^r + (1 - \xi)V(a = 0) \]
\[ \Rightarrow B^e = B^e_* = \frac{1}{1 - \xi} \left[ I - \left( a(1 - \tau) + B - c_1(B) + P \right)e^r \right].\]

Ditch:

\[(a(1 - \tau) + B - c_1(B) + P)e^r + (1 - \xi)V(a = 0) < I \Rightarrow B^e = 0.\]

The Safe situation occurs when the company’s capital level is non-negative at the end of a period without the help of emergency capital. When the company is not safe, we would raise emergency capital just enough at $B^e_* > 0$ to Save the company by leveling the
capital back to zero. Otherwise, we enter the Ditch situation, when it is not optimal to raise emergency capital to save the company. Then we can effectively eliminate the control variable $B^e$. Denote the cutoff between Safe and Save by $S = [a(1 - \tau) + B - c_1(B) + P]e^r$, and denote the cutoff between Save and Ditch by $D = (a(1 - \tau) + B - c_1(B) + P)e^r + (1 - \xi)V(a = 0)$.

The Bellman Equation becomes:

$$V(a, Q^{(1)}, ..., Q^{(N)}, M_{j-1}^{(1)}, ..., M_{j-1}^{(N)})$$

$$= \max_{\{q^{(i)}\}, \{\rho^{(i)}\}, B} \mathbb{E}\left\{ I_{\{t \leq S\}} \left[ P - e^{-r}I - \tau a - c_1(B) + e^{-r}V(a', Q^{(1)}, ..., Q^{(N)}, M_{j-1}^{(1)}, ..., M_{j-1}^{(N)}) \right] 
+ I_{\{S < I \leq D\}} \left[ \frac{1}{1 - \xi} \left( a(1 - \tau) + B - c_1(B) + P - e^{-r}I \right) - (a + B) \right] 
+ I_{\{I > D\}} (-a - B) \right\}$$

subject to the budget constraint:

$$a' = (a(1 - \tau) + B - c_1(B) + P)e^r - I.$$

premium function:

$$p^{(n)} = p_n(a, Q^{(1)}, ..., Q^{(N)}, M_{j-1}^{(1)}, ..., M_{j-1}^{(N)}, q^{(1)}, ..., q^{(N)}, P, B).$$

and regulation constraint

$$\rho(I) \leq (a(1 - \tau) + B - c_1(B) + P)e^r + (1 - \xi)V(a = 0)$$

2.3 Discussion

In the previous section, we proposed a general multi-period profit maximization model with a generic loss structure of a P&C insurance company. One crucial factor is the
number of development years. With a long-tailed business line, the loss structure entails multiple development years, possibly even ten years or more. With a short-tailed line, losses are usually incurred and paid in the accident year. So the generic loss structure still applies to the short-tailed line, but with only one development year. Thus, the current state of a one-DY line only depends on the capital level, and does not depend on the previous losses or exposures. In fact, the model in Bauer and Zanjani (2015b) is the special case of the general model with one development year in all business lines. Therefore, the general multi-period profit maximization model extends beyond the one-DY model, provides much more accuracy in a P&C insurer’s characteristics, and also dramatically increases the flexibility in modeling for a P&C company with a mix of long-tailed and short-tailed business lines.

3 Implementation – Two Lines and Two Development Years

In this section, we specify a loss structure for a P&C insurer with two long-tailed business lines and with two development years on each line (2L2DY). We then calibrate and solve for the model using numerical methods and present the results along with marginal cost of risk calculations. From the calculation, we derive the capital allocation for two lines and compare the allocation with conventional gradient allocation results.

3.1 Loss Structure with Two Lines and Two Development Years

The paid loss triangle is a 2x1 triangle for each line. Thus, there are six loss random variables of interest, three for each line as illustrated in figure 2:

\[ X_1 = \{X_1^{(1)}, X_1^{(2)}\} \]

\[ X_1^{(1)} \] and \( X_1^{(2)} \) are losses paid in the previous accident year \( t - 1 \). \( X_1^{(n)} \) and \( X_2^{(n)} \) are the losses to be paid in the current period \( t \), with \( X_1^{(n)} \) being the paid loss of accident
Figure 2: Loss Triangles for A Company under 2L2DY

year $t$ and $X_2^{(n)}$ being the second development year paid loss for the previous accident year $t - 1$. We assume $X_1^{(n)}$ and $X_2^{(n)}$ to have the following properties:

**Chain-Ladder:**

$$
\mathbb{E}(X_2^{(n)} | X_1^{(n)}) = (f^{(n)} - 1)X_1^{(n)}.
$$

$$
\mathbb{V}(X_2^{(n)} | X_1^{(n)}) = (\sigma^{(n)})^2 X_1^{(n)}.
$$

$f^{(n)}$ and $(\sigma^{(n)})^2$ are the chain ladder factors.

**Conditional Normal:**

$$
X_1^{(n)} | X_1^{(n)} = X_1^{(n)} \sim \mathcal{N}(\mu_1^{(n)}, (\sigma_1^{(n)})^2).
$$

$$
X_2^{(n)} | X_1^{(n)} \sim \mathcal{N}(\mu_2^{(n)}(X_1^{(n)}), (\sigma_2^{(n)})^2(X_1^{(n)})).
$$

**Linearly Correlation Between Lines**

$$
corr(X_1^{(1)}, X_1^{(2)}) = \rho
$$

The chain-ladder property is following Mack (1993). Chain ladder assumes $M_{t,j-1}^{(n)} = C_{t,j-1}^{(n)}$. Thus, the cumulative paid loss in each accident year is a Markov chain with the following conditional expectation and variance:

$$
\mathbb{E}(C_{t,j}^{(n)} | C_{t,j-1}^{(n)}) = f_{j-1}^{(n)} C_{t,j-1}^{(n)}
$$

$$
\mathbb{V}(C_{t,j}^{(n)} | C_{t,j-1}^{(n)}) = (\sigma_{j-1}^{(n)})^2 C_{t,j-1}^{(n)}.
$$
\( f_{j-1}^{(n)} \) is the \( j-1 \)th chain ladder factor and \( (\sigma_{j-1}^{(n)})^2 \) is the \( j-1 \)th chain ladder variance factor.

For our 2L2DY implementation, as \( C_1^{(n)} = X_1^{(n)} \) and \( C_2^{(n)} = X_1^{(n)} + X_2^{(n)} \)

\[
\mathbb{E}(C_2^{(n)}|X_1^{(n)}) = f(n) X_1^{(n)} \Rightarrow \mathbb{E}(X_2^{(n)}|X_1^{(n)}) = (f(n) - 1)X_1^{(n)}
\]

\[
\mathbb{V}(C_2^{(n)}|X_1^{(n)}) = (\sigma(n))^2 X_1^{(n)} \Rightarrow \mathbb{V}(X_2^{(n)}|X_1^{(n)}) = (\sigma(n))^2 X_1^{(n)}
\]

As conventional in this context, the indemnity is assumed to be proportional to the chosen exposure as well as previous exposures. Hence, the indemnity random variable is specified as

\[
I = \sum_{n=1}^{2} q'(n)X_1^{(n)} + q(n)X_2^{(n)}
\]

This linearity assumption entails that the marginal claim distribution is fixed, so that the loss distribution is homogeneous. Again, this is in line with typical assumptions and generalizations are possible.

Hence, \( I = \sum_{n=1}^{2} q'(n)X_1^{(n)} + q(n)X_2^{(n)} \) is also normal, with conditional mean and variance:

\[
\mu_I = \mathbb{E}(I|X_1) = \sum_{n=1}^{2} q'(n) \mu_1^{(n)} + (f(n) - 1)q(n)X_1^{(n)}
\]

\[
\sigma_I^2 = \mathbb{V}(I|X_1) = \sum_{n=1}^{2} (q'(n))^2 (\sigma_1^{(n)})^2 + (q(n))^2 (\sigma(n))^2 X_1^{(n)} + 2q'(1)q'(2)\rho \sigma_1^{(1)} \sigma_1^{(2)}
\]

Using the past loss realization \( X_{t1}^{(n)} \) and \( X_{t2}^{(n)} \) above the 2x1 triangle, we can estimate the chain ladder factors following Mack (1993):

\[
\hat{f}^{(n)} = \frac{\sum_{t=1}^{T-2} X_{t1}^{(n)} + X_{t2}^{(n)}}{\sum_{t=1}^{T-2} X_{t1}^{(n)}}
\]

\[
(\hat{\sigma}^{(n)})^2 = \frac{1}{T-3} \sum_{t=1}^{T-2} \left[ X_{t1}^{(n)} + X_{t2}^{(n)} \right] \left[ \frac{X_{t1}^{(n)} + X_{t2}^{(n)}}{X_{t1}^{(n)}} - \hat{f}^{(n)} \right]^2
\]
and normal parameters are estimated using the accident year losses from both lines:

\[
\hat{\mu}_1^{(n)} = \frac{1}{T-1} \sum_{t=1}^{T-1} X_{t1}^{(n)}
\]

\[
(\hat{\sigma}_1^{(n)})^2 = \frac{1}{T-2} \sum_{t=1}^{T-1} (X_{t1}^{(n)} - \hat{\mu}_1^{(n)})^2
\]

\[
\hat{\rho} = \frac{\sum_{t=1}^{T-1} (X_{t1}^{(1)} - \hat{\mu}_1^{(1)}) (X_{t1}^{(2)} - \hat{\mu}_1^{(2)})}{\sqrt{\sum_{t=1}^{T-1} (X_{t1}^{(1)} - \hat{\mu}_1^{(1)})^2} \sqrt{\sum_{t=1}^{T-1} (X_{t1}^{(2)} - \hat{\mu}_1^{(2)})^2}}
\]

### 3.2 Marginal Cost of Risk and Capital Allocation

In the general model, we have used a generic premium function without detailed specifications. Since people assess company quality via rating that reflects default probability, and increasing the scale of insurance business decreases profit margins, we can specify the premium function as the product of expected present value of future exposed losses and a markup function:

\[
p_n = \mathbb{E}\left[ e^{-r q'(n)} X_1^{(n)} + e^{-2r q'(n)} X_2^{(n)} | X_1 \right] \times \exp \left\{ \alpha_n - \beta_n \mathbb{P}(I > D) - \gamma_n \mathbb{E}(I) \right\}
\]

Denote \( \phi_I(.) \) the normal density for \( I \), \( I_a^{(i)} = \frac{\partial I}{\partial q'^{(i)}} \), \( V_a = \frac{\partial V}{\partial a} \) and \( V_q^{(i)} = \frac{\partial V}{\partial q'^{(i)}} \). Following the previous section, we can derive the marginal cost of risk for each line:

**Proposition 3.1** (Marginal Cost of Risk for 2L2DY implementation)
\[
\left( \frac{E[X'_1(i) + e^{-r} X'_2(i) | X_1] \times \exp \left( \alpha_i - \beta_i \mathbb{P}(I > D) - \gamma_i \mathbb{E}(I) \right)}{1 - c'_1(B)} \right)
\]

\[
= \mathbb{E}[I_q(i) 1_{\{I \leq D\}}] + \gamma_i \mathbb{E}[I_q(i)] \frac{Pe^r}{1 - c'_1(B)} + \mathbb{E}[I_q(i) V_q 1_{\{I \leq S\}}] + \frac{\xi}{1 - \xi} \mathbb{E}[I_q(i) 1_{\{S < I \leq D\}}]
\]

+ \left( \mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_I}{\partial q'} - (1 - \xi) V_q^{(i)}(a = 0) \right) \times (\mathbb{P}(I > D) + \tau^*) - \mathbb{E}[V_q^{(i)} 1_{\{I \leq S\}}] \]

where \( \tau^* = \frac{c'_1(B)}{1 - c'_1(B)} - \frac{\xi}{1 - \xi} \mathbb{P}(S < I \leq D) - \mathbb{E}(V_a 1_{\{I \leq S\}}) \) is the “shadow cost of capital” following Bauer and Zanjani (2015b).

The right-hand side of the equation decomposes the marginal cost of risk in line \( i \) into the following:

1. Expected actuarial cost increase when the company is solvent;
2. “Scale cost” originating from an increase of supply;
3. Expected marginal indemnity adjusted for capital holdings;
4. Expected marginal cost of saving a company;
5. Expected marginal cost related to probability of default.
6. Additional marginal cost due to the “after shock” effect of the previous exposure.

The last component raises our attention because it reflects the “after shock” feature of a P&C company. The exposure decision of the last period affects the company’s expected value in the current period, and also affects the company’s allocation decision. This component can significantly affect the allocation weight for the long-tailed business lines, as is illustrated by results in the next section.
3.3 Calibration

To calibrate the premium function for each line, we collect the net premium collected, loss paid, loss adjustment costs and underwriting costs from 2007 to 2013 NAIC data. To obtain the default probability, we use the one-year default probability in the Exhibit 2 under Best’s Impairment Rate and Rating Transition Study - 1977-2014 with proper interpolation. We then fit the following model to obtain the premium parameters:

\[ \log p_{nit} = \alpha_n + \alpha nt + \beta_n d_{nit} + \gamma_n E_{nit} + \epsilon_{nit} \]

\( p_{nit} \) is the price of insurance line \( n \), of company \( i \) in year \( t \), calculated using a ratio of net premium earned to the sum of loss paid, loss adjustment expense and underwriting expense. \( d_{nit} \) denotes the default probability. \( E_{nit} \) is the expected loss, a measure of the size of the company, calculated using the net premium earned multiplied by average loss ratio. \( \epsilon_{nit} \) is the error term. Table 1 shows the resulted premium parameters for each line.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Worker’s Compensation</th>
<th></th>
<th>Commercial Auto</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>Std. Error</td>
<td>t-value</td>
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<tr>
<td>( \alpha )</td>
<td>0.0984</td>
<td>0.0432</td>
<td>2.28</td>
</tr>
<tr>
<td>( \beta )</td>
<td>-5.4043</td>
<td>2.3134</td>
<td>-2.34</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>-6.05e11</td>
<td>3.99e-11</td>
<td>-1.52</td>
</tr>
<tr>
<td>Observations: 389. Adj. ( R^2 ): 0.0117</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Estimated Premium Parameters

To numerically solve the Bellman equation, we use the paid loss triangles from one P&C insurance company, obtained from NAIC Schedule P data from 2013. Table 2 presents the full triangles for paid losses (in thousands) in the workers’ compensation line and commercial auto line. To be used for the 2L2DY model, losses beyond the second
development year are discounted back using corresponding yield curves from each accident year. Then we estimate the chain-ladder and normal parameters, presented in table 3. We set the capital costs as $\tau = 3\%$, $c_1 = 12.5\%$, $\xi = 0.70$. The interest rate of asset investment is $r = 3\%$.

<table>
<thead>
<tr>
<th>AY</th>
<th>DY</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<th>9</th>
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<table>
<thead>
<tr>
<th>AY</th>
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<th>4</th>
<th>5</th>
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<td>5,729</td>
<td>5,883</td>
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<tr>
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<tr>
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<td>16,791</td>
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<tr>
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</tbody>
</table>

Table 2: Cumulative Paid Loss Triangles of Worker’s Compensation Line (upper) and Commercial Auto Line (lower) from A P&C Insurance Company ($000 omitted)
Table 3: Estimated Parameter Values for Loss Triangles

<table>
<thead>
<tr>
<th></th>
<th>Worker’s Comp</th>
<th>Comm Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>3.38</td>
<td>3.74</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>5.45e6</td>
<td>5.46e6</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>2.22e7</td>
<td>1.29e7</td>
</tr>
<tr>
<td>$(\sigma_1)^2$</td>
<td>2.01e14</td>
<td>5.18e13</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.98</td>
<td></td>
</tr>
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</table>

The estimated parameters suggest that the company has a larger workers’ compensation business than the commercial auto. The losses paid in the accident year are highly correlated. Both workers’ compensation and the commercial auto line are considered long-tailed. This is in line with our estimation results, reflected by the estimate of chain ladder factors $f$ associated with both lines.

3.4 Results

The results vary in the state-space. Different combinations of last-period exposure and loss realization yield distinct optimal policies and marginal cost results. The capital and exposure state variables are "controllable" by the insurance company through its policy, while the loss state variables have a stochastic nature and are "uncontrollable" by the company. To make sense of the results, we will demonstrate them as how the value, policy and marginal costs behave under "controllable" states, subject to "uncontrollable" hits by the loss realization states. We define the "uncontrollable" hits of five loss states—average loss, small loss on both, large loss on both, large loss on worker’s compensation, and large loss on commercial auto—in the following table:
The optimal policies of the company are summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Average Loss</th>
<th>Small Loss on Both</th>
<th>Large Loss on Both</th>
<th>Large Loss on W Comp</th>
<th>Large Loss on C Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td>W Comp $X_1^{(1)}$</td>
<td>$\mu_1^{(1)}$</td>
<td>$\mu_1^{(1)} - \sigma_1^{(1)}$</td>
<td>$\mu_1^{(1)} + \sigma_1^{(1)}$</td>
<td>$\mu_1^{(1)} + \sigma_1^{(1)}$</td>
<td>$\mu_1^{(1)}$</td>
</tr>
<tr>
<td>C Auto $X_1^{(2)}$</td>
<td>$\mu_1^{(2)}$</td>
<td>$\mu_1^{(2)} - \sigma_1^{(2)}$</td>
<td>$\mu_1^{(2)} + \sigma_1^{(2)}$</td>
<td>$\mu_1^{(2)}$</td>
<td>$\mu_1^{(2)} + \sigma_1^{(2)}$</td>
</tr>
</tbody>
</table>

Table 4: Definition of Loss States

"Zero capital" means that the company sheds all capital that it holds, or the optimal $B^* = -a(1-\tau)$. "Positive capital" means that the company would keep a positive amount of capital, or the optimal $B^* > -a(1-\tau)$. "Full exposure" means that the company writes as much insurance as it can, while not as much under "Limited exposure". Figure 3 illustrates the "zero capital" and "full exposure" and the value function under this policy. Figure 4 illustrates the "positive capital" and "limited exposure" and the value function under this policy. The Figure 4, in fact, shows the combination of two policies under large last-period loss realization on both lines. When the company has small last-period exposure in workers’ compensation, the optimal policy is "zero capital" and "full exposure", corresponding to the left part of figure 4b and figure 4c. When the company has large last-period exposure in workers’ compensation, the optimal policies become "positive capital" and "limited exposure", corresponding to the right part of figure 4b and figure 4c. The
value function of figure 4a changes as the last-period exposure to workers’ compensation increases. The change is illustrated in figure 5. When the company has a high exposure on workers’ compensation in the last period, the value function with respect to capital increases until the optimal capital level, and then decreases, corresponding to the left two graphs in figure 5. When the company has low exposure on workers’ compensation in the last period, the value function is decreasing with respect to capital, corresponding to the right two graphs in figure 5.

Under most situations, the optimal policy for the company is to keep zero capital and full exposure. It is worth noting that when the company is in the worse scenario and hit by large loss on both lines, as long as it does not have high last-period exposure on the bigger workers’ compensation line, the company still adopts the zero capital and full exposure policy.

The marginal cost calculations are shown in table 6 to table 9. Each table presents the each component of marginal cost under each loss state, given a combination of last-period exposure level. It is worth noting that wherever the default probability is non-zero, the company always holds positive capital as the optimal policy. The after-shock effect term in the capital allocation equation turns out to be a significant part of the marginal cost besides the capital cost. In figure 3a and figure 4a, it is clear that the marginal effect of exposure having on the value function is much more than the marginal effect of capital. The marginal cost calculations shed some lights on the relationship between loss development and capital allocation. First, a larger last-period loss realization shifts the allocation weight from the line with smaller chain-ladder factor (workers’ compensation line) to the line with larger chain-ladder factor (commercial auto line). This phenomenon is best explained by the 6th component in the marginal cost equation (7), which is the additional marginal cost due to the “after shock” effect of the previous exposure. Exploration of the
relationship between increase in allocation weight of longer-tailed lines and increase in last-period loss realization could lead to a future empirical study.

![Figure 3: Value Function, External Capital Raising and Exposure Decision under "Zero Capital" and "Full Exposure", Small Last-Period Loss Realization](image)

(a) Value  
(b) External Capital  
(c) Exposure

Figure 3: Value Function, External Capital Raising and Exposure Decision under "Zero Capital" and "Full Exposure", Small Last-Period Loss Realization

![Figure 4: Value Function, External Capital Raising, and Exposure Decision under "Positive Capital" and "Limited Exposure", Large Last-Period Loss Realization](image)

(a) Value  
(b) External Capital  
(c) Exposure

Figure 4: Value Function, External Capital Raising, and Exposure Decision under "Positive Capital" and "Limited Exposure", Large Last-Period Loss Realization
4 Conclusion

The model presented in section 2 takes into account the loss structure of a P&C insurance company, which is missing in the previous literature. The general model is very flexible and can be applied to insurance companies that have both short-tailed and long-tailed business lines. The implementation of the model with two lines and two development years shows the results of capital allocation vary depending on previous loss exposure and loss realization. Various extensions are possible. First, for tractability, we adopt the chain-ladder method with normal Markov chain assumption in the implementation. As many actuarial literatures have explored, there are more advanced models for estimating
<table>
<thead>
<tr>
<th>Marginal Cost Component</th>
<th>Average Loss</th>
<th>Small Loss on Both</th>
<th>Large Loss on Both</th>
<th>Large Loss on W Comp</th>
<th>Large Loss on C Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
</tr>
<tr>
<td>Actuarial Cost</td>
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<td>22,199,296</td>
<td>12,861,000</td>
<td>22,209,812</td>
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<td>60,400</td>
<td>1,027,243</td>
<td>60,382</td>
<td>1,026,498</td>
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<tr>
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<td>-385,818</td>
<td>-666,006</td>
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<td>-666,114</td>
</tr>
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<td>Capital Cost</td>
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<td>39,509,996</td>
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<td>59,672,193</td>
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<tr>
<td>After Shock Effect</td>
<td>79,253,032</td>
<td>36,220,338</td>
<td>24,289,265</td>
<td>20,490,949</td>
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<td>Total</td>
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<td>86,375,982</td>
<td>53,688,585</td>
<td>484,697,671</td>
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<tr>
<td>Allocation %</td>
<td>67.25%</td>
<td>32.75%</td>
<td>61.67%</td>
<td>38.33%</td>
<td>63.89%</td>
</tr>
</tbody>
</table>

"Shadow Cost" $\tau^*$
- $0.17222$
- $0.17225$
- $0.17243$
- $0.17238$
- $0.17232$

Default Probability
- $0$
- $0$
- $0$
- $0$
- $0$

Optimal Capital $a$
- $0$
- $0$
- $0$
- $0$
- $0$

Table 6: Marginal Costs of Risk under Low Last-Period Exposure on Both Lines
<table>
<thead>
<tr>
<th>Component</th>
<th>Average Loss</th>
<th>Small Loss on Both</th>
<th>Large Loss on Both</th>
<th>Large Loss on W Comp</th>
<th>Large Loss on C Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
</tr>
<tr>
<td>Actuarial Cost</td>
<td>22,206,025</td>
<td>12,863,273</td>
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<td>1,005,755</td>
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<td>-665,424</td>
<td>-385,371</td>
<td>542,333</td>
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<td>Saving Cost</td>
<td>3,672,289</td>
<td>1,864,199</td>
<td>64,698</td>
<td>40,474</td>
<td>7,567,817</td>
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<td>102,086</td>
<td>35,115,289</td>
<td>18,469,588</td>
<td>-95,917,408</td>
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<td>After Shock Effect</td>
<td>230,234,273</td>
<td>177,308,110</td>
<td>107,210,243</td>
<td>63,163,725</td>
<td>636,468,118</td>
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<td>Total</td>
<td>256,673,962</td>
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<td>164,950,768</td>
<td>94,210,560</td>
<td>571,840,492</td>
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<td>Allocation %</td>
<td>57.23%</td>
<td>42.77%</td>
<td>63.65%</td>
<td>36.35%</td>
<td>57.25%</td>
</tr>
</tbody>
</table>

| "Shadow Cost" τ*   | 0.00095      | 0.17111            | -0.32515           | -0.33008             | -0.02402           |
| Default Probability | 0            | 0                  | 0.089%             | 0.049%               | 0                  |
| Optimal Capital a   | 0            | 0                  | 200,000,000        | 100,000,000          | 0                  |

Table 7: Marginal Costs of Risk under High Last-Period Exposure on Workers’ Compensation Line
<table>
<thead>
<tr>
<th>Component</th>
<th>Average Loss</th>
<th>Small Loss on Both</th>
<th>Large Loss on Both</th>
<th>Large Loss on W Comp</th>
<th>Large Loss on C Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
</tr>
<tr>
<td>Actuarial Cost</td>
<td>22,194,584</td>
<td>12,856,909</td>
<td>22,203,390</td>
<td>12,864,039</td>
<td>22,198,949</td>
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<td>60,300</td>
<td>1,016,770</td>
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<td>Continuation Effect</td>
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<td>-665,741</td>
<td>-385,512</td>
<td>-595,037</td>
</tr>
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<td>Saving Cost</td>
<td>1,008,109</td>
<td>525,586</td>
<td>47,540</td>
<td>12,882</td>
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<td>Capital Cost</td>
<td>29,961,401</td>
<td>17,543,827</td>
<td>34,508,582</td>
<td>20,277,806</td>
<td>-28,080,929</td>
</tr>
<tr>
<td>After Shock Effect</td>
<td>300,351,600</td>
<td>203,554,603</td>
<td>61,287,193</td>
<td>20,277,806</td>
<td>493,467,958</td>
</tr>
<tr>
<td>Total</td>
<td>353,883,705</td>
<td>234,161,637</td>
<td>118,406,815</td>
<td>113,226,742</td>
<td>493,532,481</td>
</tr>
</tbody>
</table>

| Allocation %               | 60.18%       | 39.82%             | 51.12%             | 48.88%               | 68.00%               | 32.00%               |

| "Shadow Cost" $\tau^*$    | 0.13856      | 0.17176            | -0.11047           | 0.12502              | -0.05692             |
| Default Probability        | 0            | 0                  | 0                  | 0                    | 0                    |
| Optimal Capital $\alpha$   | 0            | 0                  | 0                  | 0                    | 0                    |

Table 8: Marginal Costs of Risk under High Last-Period Exposure on Commercial Auto Line
Table 9: Marginal Costs of Risk under High Last-Period Exposure on Both Lines

<table>
<thead>
<tr>
<th>Marginal Cost Component</th>
<th>Average Loss</th>
<th>Small Loss on Both</th>
<th>Large Loss on Both</th>
<th>Large Loss on W Comp</th>
<th>Large Loss on C Auto</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
<td>C Auto</td>
<td>W Comp</td>
</tr>
<tr>
<td>Actuarial Cost</td>
<td>22,181,529</td>
<td>12,852,008</td>
<td>22,205,229</td>
<td>12,859,804</td>
<td>22,090,195</td>
</tr>
<tr>
<td>Scale Cost</td>
<td>1,009,912</td>
<td>59,363</td>
<td>1,022,448</td>
<td>60,100</td>
<td>994,842</td>
</tr>
<tr>
<td>Continuation Effect</td>
<td>227,473</td>
<td>116,490</td>
<td>-658,529</td>
<td>-382,135</td>
<td>1,558,009</td>
</tr>
<tr>
<td>Saving Cost</td>
<td>8,016,314</td>
<td>4,043,394</td>
<td>558,303</td>
<td>278,435</td>
<td>1,881,887</td>
</tr>
<tr>
<td>Total</td>
<td>327,498,534</td>
<td>219,887,819</td>
<td>162,801,075</td>
<td>119,975,550</td>
<td>1,022,942,901</td>
</tr>
</tbody>
</table>

| Allocation %            | 59.83% | 40.17% | 57.57% | 42.43% | 60.26% | 39.74% | 67.08% | 32.92% | 49.52% | 50.48% |

| "Shadow Cost" τ*        | -0.34833 | 0.15498 | -0.01216 | -0.09563 | -0.33379 |
| Default Probability     | 0.029% | 0 | 0.404% | 0.350% | 0.067% |
| Optimal Capital α       | 100,000,000 | 0 | 300,000,000 | 300,000,000 | 100,000,000 |
and forecasting loss triangles. Second, we begin with the assumption that all the assets are invested at a fixed interest rate. Adding the security market volatility into the general model would further bridge the gap between model and reality.

**Appendix A: Proofs of the Propositions**

*Proof of Proposition 2.1:* In the spirit of Bertsekas (1995), the infinite horizon optimization problem (4) subject to (1) is exactly resulting in the Bellman equation (6).

*Proof of Proposition 2.2:* We are looking for $B^e \in \{0, B^e_*\}$ such that

$$I - (a(1 - \tau) + B - c_1(B) + P)e^r = B^e - c_2(B^e)$$

And given the linear cost $c_2(x) = \xi x$, $B^e_*$ can be solved

$$B^e_* = \frac{1}{1 - \xi} \left[ I - \left( a(1 - \tau) + B - c_1(B) + P \right)e^r \right]$$

(i) When $a' = (a(1 - \tau) + B - c_1(B) + P)e^r + B^e - c_2(B^e) > I$, and if $B^e > 0$, then by reducing $B^e$ by a small amount $\epsilon < a'$ (so the company remains solvent) and increasing $B$ in the next period by $\epsilon$, we gain from the cost difference because $c'_1(\cdot) < c'_2(\cdot)$. Thus, raising any positive amount of $B^e$ is not optimal. Hence, $B^e = 0$.

(ii) When $a' = (a(1 - \tau) + B - c_1(B) + P)e^r + B^e - c_2(B^e) < I$, the objective function is decreasing with respect to $B^e$. Thus, the optimal $B^e = 0$.

(iii) When $a' = (a(1 - \tau) + B - c_1(B) + P)e^r + B^e - c_2(B^e) = I$, $B^e = B^e_*$. Furthermore, when $(a(1 - \tau) + B - c_1(B) + P)e^r \geq I$, it entirely falls in (i), so $B^e = 0$.

When $(a(1 - \tau) + B - c_1(B) + P)e^r < I$, raise at most $B^e = B^e_*$ to reach zero-capital level (iii). It is optimal to save the company if the net present value of saving, i.e.

$$P - e^{-r}I - \tau a - c_1(B) - e^{-r}c_2(B^e_*) + e^{-r}V\left((a(1 - \tau) + B - c_1(B) + P)e^r + B^e_* - c_2(B^e_*) - I\right) =$$
\[ P - e^{-r}I - \tau a - c_1(B) - e^{-r}c_2(B^e) + e^{-r}V(a = 0) \] is not less than the net present value of ditching, i.e. \(-a + B\). Solve the following inequation:

\[ P - e^{-r}I - \tau a - c_1(B) - e^{-r}c_2(B^e) + e^{-r}V(a = 0) \geq -a + B \]

The result is \(V(a = 0) \geq B^e\), or

\[ I \leq \left(a(1 - \tau) + B - c_1(B) + P\right)e^r + (1 - \xi)V(a = 0) \]

when it is optimal to raise \(B^e = B^e\) to save the company. Otherwise if \(V(a = 0) < B^e\), it is not optimal to raise the emergency capital. Thus when \(V(a = 0) < B^e\), or equivalently \(I > \left(a(1 - \tau) + B - c_1(B) + P\right)e^r + (1 - \xi)V(a = 0)\), \(B^e = 0\).

**Proof of Proposition 3.1:** Derive expressions for marginal costs from Lagrangians of the right-hand size of the Bellman equation:

\[
\mathcal{L} = \mathbb{E}\left\{ I_{\{I \leq S\}} \left[ P - e^{-r}I - \tau a - c_1(B) + e^{-r}V(a', Q',...,Q'_{M(N)}, M_{j-1}',...,M'_{j-1}) \right] \\
+ I_{\{S < I \leq D\}} \left[ \frac{1}{1 - \xi} \left(a(1 - \tau) + B - c_1(B) + P - e^{-r}I\right) - (a + B) \right] + I_{\{I > D\}}(-a - B) \right\} \\
- \sum_{n=1}^{N} \lambda_n \left(p^{(n)} - p_n(a, Q^{(1)},...,Q^{(N)}, M_{j-1}',...,M'_{j-1}, q^{(1)},...,q^{(N)}, P, B)\right) \\
- \eta \left(e^{-r}\rho(I) - (a(1 - \tau) + B - c_1(B) + P) - e^{-r}(1 - \xi)V(a = 0)\right) \\
\]  

Denote \(I_q^{(i)} = \frac{\partial I}{\partial q^{(i)}}, I_p^{(i)} = \frac{\partial I}{\partial p^{(i)}}, V_a = \frac{\partial V}{\partial a} \) and \(V_q^{(i)} = \frac{\partial V}{\partial q^{(i)}}\). The first order condition for \(q, P\) and \(B\) are

\[
\frac{\partial \mathcal{L}}{\partial q^{(i)}} = e^{-r}\mathbb{E}\left[ ( -I_q^{(i)}(1 + V_a) + V_q^{(i)} ) I_{\{I \leq S\}} - \frac{I_q^{(i)}}{1 - \xi} I_{\{S < I \leq D\}} \right] \\
+ \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial q^{(i)}} - \eta e^{-r} \left( \frac{\partial \rho}{\partial q^{(i)}} - (1 - \xi)V_q^{(i)}(a = 0) \right) = 0 \]  

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\[ \frac{\partial L}{\partial p^{(i)}} = \mathbb{E} \left[ (1 + V_a)1_{\{I \leq S\}} + \frac{1}{1 - \xi} 1_{\{S < I \leq D\}} \right] - \lambda_i + \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial p^{(i)}} + \eta = 0 \quad (10) \]

\[ \frac{\partial L}{\partial B} = \mathbb{E} \left[ (-c_1'(B) + V_a(1 - c_1'(B)))1_{\{I \leq S\}} + \left( \frac{\xi - c_1'(B)}{1 - \xi} \right) 1_{\{S < I \leq D\}} - 1_{\{I > D\}} \right] \]

\[ + \sum_{n=1}^{N} \lambda_n \frac{\partial p_n}{\partial B} + \eta(1 - c_1'(B)) = 0 \quad (11) \]

From (8) and (9), we can solve for lambda vector \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \). Denote

\[ E_q = \left( \text{an N x 1 vector of } e^{-r} \mathbb{E} \left[ (1 + V_a)1_{\{I \leq S\}} - \frac{L_q^{(i)}}{1 - \xi} 1_{\{S < I \leq D\}} \right] , i = 1, 2, ..., N \right) , \]

\[ E_{PB} = \left( \text{an N x 1 vector of } (N - 1) \mathbb{E} \left[ (1 + V_a)1_{\{I \leq S\}} + \frac{1}{1 - \xi} 1_{\{S < I \leq D\}} \right] \right) \]

and

\[ \mathbb{E} \left[ (-c_1'(B) + V_a(1 - c_1'(B)))1_{\{I \leq S\}} + \left( \frac{\xi - c_1'(B)}{1 - \xi} \right) 1_{\{S < I \leq D\}} - 1_{\{I > D\}} \right] . \]

Denote the following matrices:

\[ \Lambda_q = \begin{pmatrix}
\frac{\partial p_1}{\partial q^{(1)}} & \frac{\partial p_2}{\partial q^{(1)}} & \cdots & \frac{\partial p_N}{\partial q^{(1)}} \\
\frac{\partial p_1}{\partial q^{(2)}} & \cdots & \cdots & \cdots \\
\frac{\partial p_1}{\partial q^{(3)}} & \cdots & \cdots & \cdots \\
\frac{\partial p_1}{\partial q^{(N)}} & \cdots & \cdots & \frac{\partial p_N}{\partial q^{(N)}}
\end{pmatrix} \]

\[ \Lambda_{PB} = \begin{pmatrix}
1 & \frac{\partial p_2}{\partial p^{(1)}} & \cdots & \frac{\partial p_N}{\partial p^{(1)}} \\
\frac{\partial p_1}{\partial p^{(2)}} & 1 & \cdots & \cdots \\
\frac{\partial p_1}{\partial p^{(N-1)}} & \cdots & 1 & \cdots \\
\frac{\partial p_1}{\partial B} & \frac{\partial p_2}{\partial B} & \cdots & \frac{\partial p_N}{\partial B}
\end{pmatrix} \]
\[ \Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \]

Solve for \( \Lambda = \Lambda_{PB}^{-1}E_{PB} \), then use \( \Lambda_q \Lambda = E_q \) to obtain capital allocation equation. A special capital allocation equation in 2L2DY case can be derived as follows:

From the premium function,

\[
p_n = \mathbb{E} \left[ e^{-r_q(n)}X_1^{(n)} + e^{-2r_q(n)}X_2^{(n)} \right| X_1] \times \exp \left\{ \alpha_n - \beta_n \mathbb{P}(I > D) - \gamma_n \mathbb{E}(I) \right\}
\]

We can derive the following derivatives for \( p_n \) w.r.t. \( q'(i) \), \( p^{(i)} \) and \( B \). Denote \( \phi(.) \) as normal density.

For \( n = i \)

\[
\frac{\partial p_i}{\partial q'(i)} = \mathbb{E} \left[ e^{-r}X_1^{(i)} + e^{-2r}X_2^{(i)} \right| X_1] \times \exp \left( \alpha_i - \beta_i \mathbb{P}(I > D) - \gamma_i \mathbb{E}(I) \right) + p_i \left[ \beta_i \phi(D) \left( (1 - \xi) I^{(i)}(a = 0) - \left( \mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_I}{\partial q'(i)} \right) \right) - \gamma_i \mathbb{E}(I_q^{(i)}) \right]
\]

For \( n \neq i \)

\[
\frac{\partial p_n}{\partial q'(i)} = p_n \left[ \beta_n \phi(D) \left( (1 - \xi) I^{(i)}(a = 0) - \left( \mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_I}{\partial q'(i)} \right) \right) - \gamma_n \mathbb{E}(I_q^{(i)}) \right]
\]

And

\[
\frac{\partial p_n}{\partial p^{(i)}} = p_n \beta_n e^r \phi(D)
\]

\[
\frac{\partial p_n}{\partial B} = p_n \beta_n e^r (1 - c_1(B)) \phi(D)
\]
Then, derive the first order conditions for the \( i \)th line (the other line denote as \( k \)):

\[
\frac{\partial L}{\partial q^{(i)}} = e^{-r} \mathbb{E} \left[ (-I_q^{(i)}(1 + V_0) + V_q^{(i)})(I_{t \leq S} - \frac{I_q^{(i)}}{1 - \xi} I_{t < I \leq D}) \right] \\
+ \lambda_i \mathbb{E} \left[ e^{-r} X_1^{(i)} + e^{-2r} X_2^{(i)}|X_1| \right] \times \exp \left( \alpha_i - \beta_i \mathbb{P}(I > D) - \gamma_i \mathbb{E}(I) \right) \\
+ \lambda_i p_i \left[ \beta_i \phi_I(D) \left( (1 - \xi) V_q^{(i)}(a = 0) - \left( \mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_I}{\partial q^{(i)}} \right) \right) \right] \\
+ \lambda_i p_k \left[ \beta_k \phi_I(D) \left( (1 - \xi) V_q^{(i)}(a = 0) - \left( \mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_I}{\partial q^{(i)}} \right) \right) \right] \\
- \eta e^{-r} \left( \frac{\partial p}{\partial q^{(i)}} - (1 - \xi) V_q^{(i)}(a = 0) \right) = 0
\]

\[ (12) \]

\[
\frac{\partial L}{\partial p^{(i)}} = \mathbb{E} \left[ (1 + V_0) I_{t \leq S} + \frac{1}{1 - \xi} I_{t < I \leq D} \right] \\
- \lambda_i + \lambda_i p_i \beta e^r \phi_I(D) + \lambda_k p_k \beta e^r \phi_I(D) + \eta = 0 \\
\Rightarrow \mathbb{E} \left[ V_a I_{t \leq S} \right] + \mathbb{P}(I \leq S) + \frac{1}{1 - \xi} \mathbb{P}(S < I \leq D) \\
- \lambda_i + \lambda_i p_i \beta e^r \phi_I(D) + \lambda_k p_k \beta e^r \phi_I(D) + \eta = 0
\]

\[ (13) \]

\[
\frac{\partial L}{\partial B} = \mathbb{E} \left[ (-c'_1(B) + V_a(1 - c'_1(B))) I_{t \leq S} + \left( \frac{\xi - c'_1(B)}{1 - \xi} \right) I_{t < I \leq D} - I_{t > D} \right] \\
+ \lambda_i p_i \beta e^r \phi_I(D)(1 - c'_1(B)) + \lambda_k p_k \beta e^r \phi_I(D)(1 - c'_1(B)) + \eta(1 - c'_1(B)) = 0 \\
\Rightarrow \mathbb{E} \left[ V_a I_{t \leq S} \right] + \mathbb{P}(I \leq S) + \frac{1}{1 - \xi} \mathbb{P}(S < I \leq D) - \frac{1}{1 - c'_1(B)} \\
+ \lambda_i p_i \beta e^r \phi_I(D) + \lambda_k p_k \beta e^r \phi_I(D) + \eta = 0
\]

\[ (14) \]

Using (13) and (14), we can solve for \( \lambda_i \)

\[
\lambda_i = \frac{1}{1 - c'_1(B)}
\]

\[ (15) \]

Using (14) and (15), we can solve for \( \eta \)
\[ \eta = \mathbb{P}(I > D) + \frac{c_1'(B)}{1 - c_1'(B)} \frac{\xi}{1 - \xi} \mathbb{P}(S < I \leq D) - \mathbb{E}(V_a I_{I \leq S}) - \frac{e^r \phi_I(D)(p_i \beta_i + p_k \beta_k)}{1 - c_1'(B)} \]  

(16)

Without regulation constraint, \( \eta = 0 \), thus:

\[ \frac{e^r \phi_I(D)(p_i \beta_i + p_k \beta_k)}{1 - c_1'(B)} = \mathbb{P}(I > D) + \frac{c_1'(B)}{1 - c_1'(B)} \frac{\xi}{1 - \xi} \mathbb{P}(S < I \leq D) - \mathbb{E}(V_a I_{I \leq S}) \]

Using (12), (15), and (16):

\[
\frac{\left( \mathbb{E}\left[X_1^{i(i)} + e^{-r} X_2^{i(i)}|X_1\right] \times \exp\left(\alpha_i - \beta_i \mathbb{P}(I > D) - \gamma_i \mathbb{E}(I)\right) \right)}{1 - c_1'(B)}
\]

\[
= \mathbb{E}\left[I_q^{(i)} I_{I \leq D}\right] + \gamma_i \mathbb{E}\left[I_q^{(i)}\right] \frac{P e^r}{1 - c_1'(B)} + \mathbb{E}\left[I_q^{(i)} V_a I_{I \leq S}\right] + \frac{\xi}{1 - \xi} \mathbb{E}\left[I_q^{(i)} I_{S < I \leq D}\right] + \left(\mu_1^{(i)} + \frac{D - \mu_I}{\sigma_I} \frac{\partial \sigma_i}{\partial q^{(i)}} - (1 - \xi)V_q^{(i)}(a = 0)\right) \times \left( \mathbb{P}(I > D) + \frac{c_1'(B)}{1 - c_1'(B)} \frac{\xi}{1 - \xi} \mathbb{P}(S < I \leq D) - \mathbb{E}(V_a I_{I \leq S}) \right) - \mathbb{E}\left[V_q^{(i)} I_{I \leq S}\right]
\]

\square

**Appendix B: Numerical implementation of the 2L2DY Model**

Because of the nature of the value function is unknown, we need to use the value iteration method to solve the Bellman equation (7) on a discretized state-space. For 2L2DY model, there are five state variables: capital \( a \), last-period exposures \( q^{(1)} \) and \( q^{(2)} \), and last-period loss realizations \( X_1^{(1)} \) and \( X_1^{(2)} \). Compared to previous models without considering DY, which only has one state variable capital , the model with DY is destined to suffer from
“the curse of dimensionality”. As the general nLjDY model has hundreds, million, or even trillion times more states, each iteration of value function would take proportional more time to complete, resulting the value iteration to finish in months or even years. Solving this high-dimensional problem seems infeasible even five years ago, but thanks to the power of modern day computing, it is feasible under proper assumptions. We start with solving 2L2DY, which has the least dimension in the general nLjDY model. The value iteration is coded and run in Julia, which, in this specific numerical task, is six times faster than the popular high-level language such as R and MatLab. According to Julia language developers, Julia is a high-performance language suitable for dynamic programming and its syntax is easily adapted from R or Matlab. Julia’s high efficiency helps shorten the runtime from one week that would have taken on R, to just over one day. In the future, we will continue to refine the algorithm and implement parallel computing to further shorten the running time.

Here is the steps of solving the Bellman equation using value iteration.

1. Pick grids for \( a = (a_1, a_2, \ldots, a_m) \), \( q^{(1)} = (q_{11}, q_{12}, \ldots, q_{1n}) \), \( q^{(2)} = (q_{21}, q_{22}, \ldots, q_{2n}) \), \( X^{(1)}_1 = (x_{11}, x_{12}, \ldots, x_{1p}) \) and \( X^{(2)}_1 = (x_{21}, x_{22}, \ldots, x_{2p}) \). Set \( V_0 = g_0(a, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) \), where \( g_0 \) is an arbitrary function.

2. From the loss triangle in the Excel sheet, discount all losses beyond the second development year back to the second development year, as to simplify the loss representation to 2x1 runoff triangle.

3. Solve the optimization problem on the right hand side of the Bellman equation and get optimized state variables \( c^* \) and yield policy function \( c = h_1((a, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1); c^*) \). Then obtain the next value function \( V_1((a, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1); h_1) \) until \( V_j \) converges.

We can obtain a simplified Bellman equation for implementation:
\[ V(a, q^{(1)}, q^{(2)}, X_1^{(1)}, X_1^{(2)}) \]
\[ = \max_{q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)}, B} \mathbb{E} \left\{ \begin{array}{c}
I_{\{I \leq S\}} \left[ P - e^{-r}I - \tau a - c_1 B + e^{-r}V(a', q^{(1)}, q^{(2)}, X_1^{(1)}, X_1^{(2)}) \right] \\
+ I_{\{S < I \leq D\}} \left[ \frac{1}{1 - \xi} (a(1 - \tau) + B(1 - c_1) + P - e^{-r}I) + e^{-r}V(a = 0) - (a + B) \right] \\
+ I_{\{I > D\}} (-a - B) \end{array} \right\} \]
\[ = \max_{q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)}, B} -e^{-r} \mathbb{E} \left\{ \begin{array}{c}
I_{\{I \leq S\}} I \\
+ e^{-r} \mathbb{E} \left\{ \begin{array}{c}
I_{\{I \leq S\}} V(a', q^{(1)}, q^{(2)}, X_1^{(1)}, X_1^{(2)}) \\
+ e^{-r} \mathbb{E} \left\{ \begin{array}{c}
I_{\{I \leq S\}} S \\
+ e^{-r} \mathbb{E} \left\{ \begin{array}{c}
I_{\{S < I \leq D\}} I \end{array} \right\} \right\} \right\} \\
- \frac{a - B + e^{-r} \text{Int}}{1 - \xi} \left\{ \begin{array}{c}
\mu_I \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) - \sigma_I \phi \left( \frac{S - \mu_I}{\sigma_I} \right) \\
- \frac{1}{1 - \xi} \left[ \mu_I \left( \Phi \left( \frac{D - \mu_I}{\sigma_I} \right) - \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right) - \sigma_I \left( \phi \left( \frac{D - \mu_I}{\sigma_I} \right) - \phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right) \right] \\
+ e^{-r} S \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) + e^{-r} \frac{\text{Int}}{1 - \xi} D \left[ \Phi \left( \frac{D - \mu_I}{\sigma_I} \right) - \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right] - a - B + e^{-r} \text{Int} \end{array} \right\} \right\} \]
\[ = \max_{q^{(1)}, q^{(2)}, p^{(1)}, p^{(2)}, B} e^{-r} \left\{ \begin{array}{c}
(S - \mu_I) \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) - \sigma_I \phi \left( \frac{S - \mu_I}{\sigma_I} \right) \\
- \frac{1}{1 - \xi} \left[ (D - \mu_I) \left( \Phi \left( \frac{D - \mu_I}{\sigma_I} \right) - \Phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right) - \sigma_I \left( \phi \left( \frac{D - \mu_I}{\sigma_I} \right) - \phi \left( \frac{S - \mu_I}{\sigma_I} \right) \right) \right] \right\} - a - B + e^{-r} \text{Int} \]

Calculating the value \( \text{Int} \) requires numerical integration since the functional form of \( V \) is unknown. To start with, Pick \((1+1)\)-point grids for \( I \), say \((y_0, y_1, ..., y_l)\), with \(-\infty < y_0 < y_1 < ... < y_l = S\) let \( \varphi_i = V(a'(y_i), q^{(1)}, q^{(2)}, X^{(1)}, X^{(2)}; y_i)\). Already we
know that $q^{(1)}$ and $q^{(2)}$ are the policy, thus fixed, and use the same grid for $X^{(1)}_1$ and $X^{(2)}_1$. Unfortunately, $a'$ may not exactly fall on the original grid point of $a$. To get values of $\varphi_i$, we need to use linear interpolation on the grid of $a$. Thus, we have:

For $a'(y_i) \in (a_k, a_{k+1})$, we have

$$\varphi_i = V(a_k, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) + \frac{a'(y_i) - a_k}{a_{k+1} - a_k} \left( V(a_{k+1}, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) - V(a_k, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) \right)$$

If $a'(y_i) > a_l$, we can extrapolate it:

$$\varphi_i = V(a_l, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) + \frac{a'(y_i) - a_l}{a_l - a_{l-1}} \left( V(a_{l-1}, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) - V(a_l, q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1) \right)$$

The linear interpolation w.r.t. $I$ is

$$V(a'(y), q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1; y) = \sum_{k=0}^{l-1} \left( \varphi_k + \frac{y - y_k}{y_{k+1} - y_k} \left( \varphi_{k+1} - \varphi_k \right) \right) I[y_k, y_{k+1}](y)$$

$$\mathbb{E}\left\{ I_{\{I \leq S\}} V(a'(y), q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1; y) | X_1 \right\} = \int_{y_0}^S V(a'(y), q^{(1)}, q^{(2)}, X^{(1)}_1, X^{(2)}_1; y) f_{I|X_1}(y) dy$$

$$= \sum_{k=0}^{l-1} \left[ \left( \varphi_k - \frac{y_k \left( \varphi_{k+1} - \varphi_k \right)}{y_{k+1} - y_k} \right) \int_{y_k}^{y_{k+1}} f_{I|X_1}(y) dy + \left( \frac{\varphi_{k+1} - \varphi_k}{y_{k+1} - y_k} \right) \int_{y_k}^{y_{k+1}} y f_{I|X_1}(y) dy \right]$$

$$= I nt \left( \frac{1}{\sqrt{2\pi\sigma_I}} e^{-\frac{(y-\mu_I)^2}{2\sigma_I^2}} \right)$$

Now both integrals can be calculated in a close form because of the conditional normality of $I|X_1$, or

$$f_{I|X_1}(y) = \frac{1}{\sqrt{2\pi\sigma_I}} e^{-\frac{(y-\mu_I)^2}{2\sigma_I^2}}$$
Denote $\Phi$ is the CDF of the standard normal distribution, and $\phi$ is the PDF of the standard normal distribution. We have:

$$\int_{y_k}^{y_{k+1}} f_{I|X_1}(y) dy = \Phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \Phi\left(\frac{y_k - \mu_I}{\sigma_I}\right)$$

$$\int_{y_k}^{y_{k+1}} y f_{I|X_1}(y) dy = \mu_I \left(\Phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \Phi\left(\frac{y_k - \mu_I}{\sigma_I}\right)\right)$$

$$-\sigma_I \left(\phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \phi\left(\frac{y_k - \mu_I}{\sigma_I}\right)\right)$$

As a result, we have the Bellman Equation for 2L2DY implementation

$$V(a, q^{(1)}, q^{(2)}, X_1^{(1)}, X_1^{(2)}) = \max_{q^{(1)}, q^{(2)}, \xi, \eta, \nu, \omega, \beta} e^{-r} \left\{ (S - \mu_I) \Phi\left(\frac{S - \mu_I}{\sigma_I}\right) - \sigma_I \phi\left(\frac{S - \mu_I}{\sigma_I}\right) \right\} - a - B + e^{-r} \text{Int}$$

$$\frac{e^{-r}}{1 - \xi} \left\{ (D - \mu_I) \left[ \Phi\left(\frac{D - \mu_I}{\sigma_I}\right) - \Phi\left(\frac{S - \mu_I}{\sigma_I}\right) \right] - \sigma_I \left[ \phi\left(\frac{D - \mu_I}{\sigma_I}\right) - \phi\left(\frac{S - \mu_I}{\sigma_I}\right) \right] \right\}$$

$$\text{Int} = \sum_{k=0}^{l-1} \left\{ \phi_k \left[ \frac{y_k (\varphi_{k+1} - \varphi_k)}{y_{k+1} - y_k} \right] \left[ \Phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \Phi\left(\frac{y_k - \mu_I}{\sigma_I}\right) \right] + \left[ \varphi_{k+1} \right] \left[ \phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \phi\left(\frac{y_k - \mu_I}{\sigma_I}\right) \right] \right\}$$

$$-\sigma_I \left(\phi\left(\frac{y_{k+1} - \mu_I}{\sigma_I}\right) - \phi\left(\frac{y_k - \mu_I}{\sigma_I}\right)\right)$$

References


